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CROSSOVER SCALE OF TGE FIXED POINT WITH REPLICA SYMMETRY BREAKING IN THE RANDOM POTTS MODEL

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We study the scaling properties of the renormalization group (RG) flows in the two-dimensional random Potts model assuming general type of the replica symmetry breaking (RSB) in the renormalized coupling matrix. It is shown that in the asymptotic regime the RG flows approach the non-trivial RSB fixed point algebraically slowly which reflects the fact that such type of t ıe fixed point is marginally stable. As a consequence, the crossover spatial scale correspor.ding to the critical regime described by this fixed point turns out to be exponentially large.

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After several years of extensive studies of the critical properties of the ferromagnetic random Potts model there still remains the controversy due to the fact that in the renormalization group (RG) approach one finds two types of the attracting fixed points, the one which is replica symmetric [1,2], and the other one in which the replica symmetry is broken[3]. In this situation, which one of the two fixed points is actually reached depends on the starting conditions for the coupling matrix in the RG equations. Recent numerical studies seems to favour the critical properties corresponding to the RSB fixed point [4]. However, since the replica symmetry breaking (RSB) fixed point is only marginally stable (the spectrum of the corresponding eigenvalues starts from zero), the crossover spatial scale corresponding to the critical regime of this fixed point may turn out to be much larger than presently accessible sizes in numerical tests. To obtain the prediction for the value of the RSB crossover scale one has to derive the asymptotic behaviour of the RG flows near the RSB fixed point, and it is this problem which we address in the present Letter.

The random q-state Potts model is described by the following lattice Hamiltonian:

$$H[\sigma] = -\sum_{<i,j>} J_{ij} V(\sigma_i, \sigma_j). \tag{1}$$

Here the spins $\{\sigma_i\}$ are taking q values; the summation goes over the nearest neighbour sites; $V(\sigma, \sigma') = 1 - \delta_{\sigma,\sigma'}$ is the spin-spin interaction potential; $\{J_{ij}\}$ are random ferromagnetic coupling constants with independent distributions characterized by the narrow width g_0 around a mean value J_0. Since the randomness is assumed to be weak ($g_0 \ll J_0$), in the critical region this system can be analised in terms of the renormalization group approach based on the conformal field theory of the unperturbed model [1]. Following the standard procedure of averaging with replicas, it can be shown (see, e.g.[2]) that in the critical point the system under consideration is described by the following continuum limit replica Hamiltonian:

$$H=\sum_{a=1}^{n}H_0^a+\sum_{a\neq b}^{n}g_{ab}\int d^2x e_a(x)e_b(x) \quad (2)$$

where H_0 is the Hamiltonian of the pure system, and the second term represents the couplings of the energy operators e_a of different replicas. Strictly speaking, after performing replica averaging one obtains the replica symmetric coupling matrix $g_{ab}=g$ (where g is proportional to g_0). However, according to a qualitative physical arguments [5] one can also consider a more general situation when the replica symmetry is spontaneously broken in the coupling matrix g_{ab} such that it has the Parisi block-like structure [6].

The renormalization group for the model (2) (with the replica symmetric g_{ab}) has been derived in the two-loop approximation in [1], and using the technique developed in [2] these RG equations can be easily generalized for an arbitrary structure of the coupling matrix g_{ab} [3]:

$$\frac{d}{d\xi}g_{ab}=\tau g_{ab}+\sum_{c=1}^{n}g_{ac}g_{cb}+g_{ab}^3-g_{ab}\sum_{c=1}^{n}g_{ac}g_{ca} \quad (3)$$

(it is assumed that $g_{aa}\equiv 0$). Here ξ is the usual RG log-scale parameter, and the small parameter of the theory $\tau=-3\epsilon$ is related to the deviation ϵ of the central charge of the conformal theory for the Potts model from the one of the Ising model. In particular, for the 3-component Potts model $\epsilon=-2/15$. In fact, the renormalization group for the Potts model is derived in terms of the ϵ-expansion technique.

The critical behaviour defined by the replica symmetric fixed point of Eq.(3) has been analyzed in all detail in [1,2,4]. Here we will concentrate on the RSB case. According to the standard technique developed in the mean-field theory of spin-glasses [6], in the limit of the continuous RSB, the matrix g_{ab} is parametrized in terms of the continuous (and monotonous) *function* $g(x)$ defined in the interval $0\leq x\leq 1$. In this case, from the general Eq.(3) one can easily derive the following RG equation for the renormalized function $g(x;\xi)$ [3]:

$$\frac{d}{d\xi}g(x;\xi)=\tau g(x;\xi)-2g(x;\xi)\overline{g}(\xi)-\int_0^x dy[g(x;\xi)-g(y;\xi)]^2+g^3(x;\xi)+g(x;\xi)\overline{g^2}(\xi), \quad (4)$$

where $\overline{g}(\xi)\equiv\int_0^1 dx g(x;\xi)$ and $\overline{g^2}(\xi)\equiv\int_0^1 dx g^2(x;\xi)$. Note, that the structure of the corresponding fixed-point equation, $\frac{d}{d\xi}g(x;\xi)=0$ coincides (with the exception of the last term in the r.h.s of Eq.(4)) with the corresponding saddle-point equation for the Parisi order parameter function in the mean-field theory of spin-glasses near the phase transition point [6]. The fixed-point solution of the Eq.(4) is:

$$g_*(x)=\begin{cases}\frac{1}{3}x & \text{at } 0\leq x\leq x_1\\ g_1 & \text{at } x_1\leq x\leq 1,\end{cases} \quad (5)$$

where $x_1=3g_1$ and the value of g_1 is defined by the equation $2g_1(1-g_1)^2=\tau$, such that in the linear order in small τ, $g_1\simeq\frac{1}{2}\tau$. The critical properties defined by this fixed point are described in detail in [2,3].

Here we would like to study the problem of crossover to the critical regime defined by the RSB fixed point (5). Usually, if one deals with the renormalization group in terms of a finite number of renormalized parameters, this problem is not so difficult. In this case, if the fixed point is stable, the RG trajectories are approaching the fixed point exponentially fast, and therefore the corresponding spatial crossover scale depends on the starting parameters according to some algebraic law (with the crossover exponent defined by the smallest in absolute value eigenvalue of the linearized RG equations in the vicinity of the fixed point).

In the present case the RG Eq. (4) describes the evolution of the *function*, which formally means that we are dealing with an infinite number of renormalized parameters, and correspondingly one finds the whole *spectrum* (infinite number) of the eigenvalues of the linearized RG equations. Moreover, it is well known from the mean field theory of spin-glasses that this spectrum starts from the $\lambda = 0$, and therefore it is not so easy to tell right away what must be the typical asymptotic decay of the perturbations near the fixed point. Although the problem of stability of the saddle-point of the type given by Eq.(5) for the corresponding mean-field SG free energy functional is already studied in all detail, and the spectrum of all the eigenvalues is well known [7], in the present problem we are facing somewhat different situation.

First, in the terminology of the stability analysis of the Parisi-type structures, the RG Eq. (4) actually represents not all possible (in the mean-field spin-glasses) deviations, but only the so-called "longitudinal" modes. This makes life much easier because, as it will be shown below (see also [7]), it makes the spectrum of the eigenvalues to be *discrete* (although still accumulating towards zero), unlike the complete spectrum in the corresponding SG problem, which is it continuous.

Second, unlike spin-glasses, where the "dynamical" behaviour of the order parameter defined by Eq.(4) makes no sense (the real microscopic spin dynamics can not be reduced to such a simple dynamical equation for the replica order parameter), here it is the asymptotic "dynamical" evolution of the Parisi function in the vicinity of the fixed point which is of the main interest.

Third, in the present problem, described by Eq.(4), we have the additional term $g(x)\overline{g^2}$, which is not present in the corresponding saddle-point equation in spin-glasses. This term appears to be irrelevant for the structure of the fixed-point solution, but it turns out to be quite relevant for the asymptotic behaviour of the deviations near the fixed point.

The linear analysis of the perturbations around the fixed point, Eq.(5), is rather simple. Substituting $g(x;\xi) = g_*(x) + \phi(x;\xi)$ into Eq.(4), in the linear order in the small deviation $\phi(x;\xi)$ we get:

$$\frac{d}{d\xi}\phi(x;\xi) = 2\int_0^1 dy K(x,y)\phi(y;\xi) + 2g_*(x)\int_0^1 dy g_*(y)\phi(y;\xi), \tag{6}$$

where

$$K(x,y) = \begin{cases} -g_*(y) & \text{for } x > y \\ -g_*(x) & \text{for } x < y. \end{cases} \tag{7}$$

It can be easily checked aposteriori that since $g_* \sim \tau$, in the leading order in $\tau \ll 1$ the last term in Eq.(6) is irrelevant, so that it will be dropped in the further analysis.

Since $K(x,y) = K(y,x)$, the operator $\hat{K}$ is hermitian and it can be diagonalized. The corresponding equation for the eigenfunctions is:

$$\hat{K}\phi^{(n)} = \lambda_n \phi^{(n)}. \tag{8}$$

Taking derivative over x two times from the above equation we obtain:

$$\frac{d^2}{dx^2}\phi^{(n)}(x) - \frac{g'_*(x)}{\lambda_n}\phi^{(n)}(x), \tag{9}$$

where, according to Eq.(5), $g'_*(x) = 1/3 = \text{const}$ for $0 \leq x < x_1$, and $g'_*(x) = 0$ for $x_1 < x \leq 1$. Thus, the solution for the eigenfunction is:

$$\phi^{(n)}(x) = \begin{cases} \sin(k_n x) & \text{for } 0 \leq x \leq x_1 \\ \phi_1 = \sin(k_n x_1) & \text{for } x_1 \leq x \leq 1, \end{cases} \tag{10}$$

where

$$k_n^2 = -\frac{1}{3\lambda_n}. \tag{11}$$

Substituting these eigenfunctions in the original Eq. (8) we obtain the equation for the eigenvalues: $k_n tg(k_n x_1) = (1 - x_1)^{-1}$. Since $x_1 \sim \tau \ll 1$ the solution of this equation gives the following spectrum of the eigenvalues:

$$\lambda_n = -\frac{3\tau^2}{4\pi^2 n^2}. \tag{12}$$

We see that the eigenvalues are accumulating towards zero. Since the characteristic decay scale of the n-th order mode is of the order of $\xi_n \sim \lambda_n^{-1} \sim n^2/\tau^2$, the more higher-order modes contains the starting function $g(x; \xi = 0)$ the slower it will decay. In the extreme case, if the starting function would be composed of all the modes with equal weight, it would not decay towards the fixed point at all. Thus, if the analysis of the stability of the fixed point would have to be restricted within the linear order, the result would be not quite conclusive. Fortunately, the actual situation appears to be more complicated, and it turns out that the linear analysis is not enough.

The problem is that besides the set of the eigenfunctions described above, there also exists the whole (infinite) spectrum of the so-called "zero-mode" functions which have the eigenvalue $\lambda_0 \equiv 0$. Coming back to the original linear equation (6) one can easily check that the zero-mode is an arbitrary function $\phi_0(x)$ such that $\phi_0(x) \equiv 0$ in the interval $0 \leq x < x_1$ and $\int_{x_1}^1 dx \phi_0(x) = 0$. Apparently, a zero-mode function is orthogonal to all the "non-zero-mode" functions, Eq.(10). Since in the linear order all the deviations which contain the zero-mode function do not decay, the analysis in the second order is needed.

Let us explicitly separate the two types of modes: $\phi(x;\xi) = \phi_1(x;\xi) + \phi_0(x;\xi)$, where the function $\phi_1(x;\xi)$ is assumed to be composed of the non-zero-mode functions (10) only. Simple calculation gives the following second-order equations for the functions $\phi_1(x;\xi)$ and $\phi_0(x;\xi)$:

$$\frac{d}{d\xi}\phi_1(x;\xi) = 2\int_0^1 dy K(x,y)\phi_1(y;\xi) - 2\phi_1(x;\xi)\overline{\phi_1}(\xi) - \int_0^x dy[\phi_1(x;\xi) - \phi_1(y;\xi)]^2 +$$

$$+ g_*(x)\overline{\phi_0^2}(\xi) \tag{13}$$

in the interval $0 \leq x \leq x_1$; and

$$\frac{d}{d\xi}\phi_0(x;\xi) = -2\phi_0(x;\xi)\phi_1(\xi) - \int_{x_1}^{x} dy[\phi_0(x;\xi) - \phi_0(y;\xi)]^2 \tag{14}$$

in the interval $x_1 < x \leq 1$. In Eq.(13) all the terms like $g_*\overline{g_*\phi_1}$, $g_*\phi_1^2$, $\phi_1\overline{g_*\phi_1}$, $g_*\overline{\phi_1^2}$ which are small in τ has been omitted; and in Eq.(14) we have introduced the notation: $\phi_1(\xi) \equiv \phi_1(x = x_1;\xi)$. Below we will show that due to mutual interference of the zero- and non-zero-modes (via the last term in Eq.(13) and the first term in Eq.(14)) the asymptotic behaviour of the solutions of the above two equations appears to be sufficiently universal.

The solution of the Eq.(14) can be found exactly for an arbitrary given function $\phi_1(\xi)$ and for an arbitrary starting function $\phi_0(x;\xi = 0)$:

$$\phi_0(x;\xi) = \int_0^x dy \frac{\gamma(\xi)\phi_0'(y;0)}{\left[1 + h(\xi)\int_{x_1}^{y} dz(z - x_1)\phi_0'(z;0)\right]^2} - \tilde{\phi}(\xi), \tag{15}$$

where

$$\gamma(\xi) = \exp\{-2\int_0^{\xi} d\zeta\phi_1(\zeta)\} \tag{16}$$

and

$$h(\xi) = \int_0^{\xi} dt\gamma(t). \tag{17}$$

Here the function $\tilde{\phi}(\xi)$ is fixed by the condition $\int_{x_1}^{1} dx\phi_0(x;\xi) = 0$, and $\phi_0'(x;0)$ is the derivative over x of a given starting function $\phi_0(x;\xi = 0)$.

The solution, Eq.(15), makes possible to derive the asymptotic (at $\xi \to \infty$) behaviour of the functions ϕ_1 and ϕ_0 without explicit solutions of Eq.(13). Let us consider three different types of possible asymptotic decays of the function ϕ_1.

1. Let us assume first that the decay of the function ϕ_1 is sufficiently fast: $\phi_1 \sim \xi^{-\alpha}$ with $\alpha > 1$ (or ϕ_1 decays exponentially fast, as it should follow from the linear analysis for the function containing finite number of the eigenfunctions, Eq.(10)). Then, from Eqs.(16) and (17) one finds: $\gamma(\xi \to \infty) \to$ const and $h(\xi \to \infty) \to (\text{const})\xi$. In this case it is easy to see from the solution (15) that the function $\phi_0(x;\xi \to \infty)$ has the step-like structure: in the narrow interval $x_1 \leq x < x_1 + \Delta(\xi)$, where $\Delta(\xi) \sim \xi^{-1/2}$, its absolute value is $\phi^{(*)} \sim \xi^{-1/2}$, whereas in the rest of the interval, $x_1 + \Delta(\xi) < x \leq 1$ its value is much smaller: $\phi^{(**)} \sim \xi^{-1}$. Therefore, for the asymptotic behaviour of the last term in eq.(13) we get: $\overline{\phi_0^2} \sim (\phi^{(*)})^2\Delta + (\phi^{(**)})^2 \sim \xi^{-3/2}$. Since the linear and the quadratic terms in Eq.(13) decay correspondingly as $\xi^{-(\alpha+1)}$ and $\xi^{-2\alpha}$, where both $(\alpha + 1)$ and 2α are greater that 3/2, the last term, containing $\overline{\phi_0^2} \sim \xi^{-3/2}$, must dominate at $\xi \to \infty$. Thus, the asymptotics $\phi_1 \sim \xi^{-\alpha}$ with $\alpha > 1$ can not take place.

2. Let us assume now that the asymptotic decay of the function ϕ_1 is slow: $\phi_1 \sim \xi^{-\alpha}$ with $\alpha < 1$. In this case from Eqs.(16) and (17) one finds: $\gamma(\xi \to \infty) \sim \sim \exp\{-(\text{const})\xi^{(1-\alpha)}\}$ and $h(\xi \to \infty) \to (\text{const})$. Therefore, according to Eq.(15), the function ϕ_0 must be exponentially small: $\phi_0 \sim \exp\{-(\text{const})\xi^{(1-\alpha)}\}$, and the last term in Eq.(13) can be neglected with respect to the other terms which depend on the function ϕ_1 only. In this case, however, if $\alpha < 1$, the second order terms, being of the order of $\xi^{-2\alpha}$, must be dominating over the linear term, which is of the order of $\xi^{-(\alpha+1)}$.

On the other hand, simple estimates show that the asymptotic solution of the Eq.(13) with the second order terms only, decay as ξ^{-1}, and it can not be $\xi^{-\alpha}$ with $\alpha < 1$. Thus, the slow asymptotics $\phi_1 \sim \xi^{-\alpha}$ with $\alpha > 1$ can not take place either.

3. It can be easily checked that the only selfconsistent asymptotic decay of the function ϕ_1 is: $\phi_1 \sim A\xi^{-1}$, where A is some constant. Indeed, using again the general solution (15) for the function ϕ_0, one can easily find that depending on the value of the constant A there can exist three different regimes:

$$\overline{\phi_0^2} \sim \begin{cases} \xi^{-4A} & \text{if } A > \frac{1}{2} \\ \frac{1}{\xi^2 (\ln\xi)^{3/2}} & \text{if } A = \frac{1}{2} \\ \xi^{-(3+2A)/2} & \text{if } A < \frac{1}{2}. \end{cases} \tag{18}$$

Coming back to Eq.(13) for the function ϕ_1, one can easily check that the last term $\overline{\phi_0^2}$ can be neglected (and only in this case the asymptotics $\phi_1 \sim \xi^{-1}$ can appear) only in the first two cases, e.i. for $A \geq 1/2$.

Thus, we conclude that in the considered RG approach with continuous RSB, for a generic starting Parisi function $g(x; \xi = 0)$ the deviations of the renormalized function $g(x;\xi)$ from the fixed point $g_*(x)$, Eq.(5), decay as ξ^{-1}. This slow asymptotic behaviour is essentially different from the usual exponentially fast decay in the vicinity of a stable fixed point in the RSB renormalization group. As the consequence, the crossover scale ξ_* which corresponds to the RSB fixed point (5) is defined by the condition $\xi_* \sim g_*^{-1} \sim \tau^{-1}$, and therefore the corresponding spatial crossover scale must be exponentially large:

$$R_* \sim \exp\{\text{const}/\tau\}. \tag{19}$$

It should be noted that the actual value of this crossover scale (which is quite important for reliable interpretation of numerical tests) essentially depends on a non-universal (const) which is defined by the structure of the starting Parisi function $g(x; \xi = 0)$. Since the structure of this function remains quite nuclear, at present stage it is hardly possible to derive more concrete prediction for the RSB crossover scale. Nevertheless, present study demonstrates that in principle this scale may appear to be well beyond the sizes accessible in the usual numerical simulations.

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