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Topological quantum mechanics for physicists

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This text is an attempt to write an introduction and the outline of main results of topological quantum mechanics for readers with the physical background. Instead of presenting rigorous mathematical formulations we concentrate on explanation of the physical ideas that underline most of constructions. We review here topological quantum mechanics since it is the simplest in the diverse family of topological theories that contains most of their common properties.

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1. Introduction. 1a. *Topological theories in modern physics.* Topological theories are almost 25 years old. They appeared as a main tool in the study of spontaneous supersymmetry breaking (Witten's index, see [1]). Another example of topological theories appeared in the nonperturbative treatment of quantum field theory in the language of vacuum condensates. It was observed [5] that in the supersymmetric theories the correlator of chiral observables (whose condensates determine nonperturbative behavior of the theory) does not depend on the distance between observable and have simple scaling behavior with respect to dilatation of the metric (later this was interpreted by Witten as a metric independence of correlators in so called twisted theories [4]).

However, it was only the beginning. Topological theories served as a worldsheet theories in the so-called topological strings. These constructions appeared in the alternative description of the noncritical strings (for $c \leq 1$) and matrix models [2], but later on they were found as a tractable subsectors inside superstrings [13]. Moreover, they turned out to be rather interesting theories themselves and finally turned out to be in the core of the most recent attempt [6] to get a deeper understanding of the quantization of the very heart of the string theory – its majesty the superstring theory!

It looks as an insolent conjecture but we cannot ignore that there are breathtaking indications [7] that topological theories can shed light on the main mystery of the last decade – the fundamental degrees of freedom of the M-theory, that is a nonperturbative theory underlying string theory.

It looks even more intriguing in the light of the recent advances in topological strings and indication of existence of the topological M-theory [8, 9] that plays the same role for topological strings as M-theory plays for superstrings.

It would be considered as a great surprise at any times but not in the recent 30 years that topological theories in different dimensions can be also considered as a clever way to pack together and relate to each other interesting numbers appearing in different areas of modern mathematics. From this point of view mathematics appears as an experimental device testing validity of the field theoretical reasoning in topological field theories. We should mention that mathematicians cannot cope effectively with the bunch of conjectures coming from physics that they have to prove in their conventional fashion.

1b. *Topological quantum mechanics as a simplest model.* It is well-known that any phenomena in theoretical physics should be studied as a function of parameters of the problem with special attention to critical values of these parameters. It is also well-known that dimension is one of the most important parameters and by studying phenomena in the minimal dimension we obtain quantitative understanding of effects that in higher dimensions may be described only qualitatively. Here the most famous example is the two dimensional field theory. It physical significance is not only in description of line objects like polymers or behavior of surfaces but mostly in creation of the tractable world of nonperturbative quantum field theories; it is clear that the dimensional transmutation in two dimensional sigma model is a baby version of the much harder problem of confinement.

The known world of topological theories contains theories of dimension 4 (Donaldson theory or topological

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Yang-Mills theory), dimension 3 (Chern-Simons theory) and 2 (Topological sigma-models), but most of phenomena appear already in dimension 1 of the space-time, i.e. in topological quantum mechanics. In this paper we will concentrate on the dependence on what physicists call the gravitational background of the field theory; in simple terms this is just distances between points of insertions of operators (in higher dimensions this includes the full geometry of the worldsheet such as the complex structure on the two-dimensional surface).

Before we proceed with explanation of topological quantum mechanics we would like to clarify several points that traditionally cause confusion in the study of topological theories.

The first confusion is the demand that the worldsheet of the 1-dimensional field theory (quantum mechanics) has to be a *smooth* 1-dimensional manifolds such as an interval, a circle or a straight line. Somehow even classically we do not study mechanics on a graphs (by the way, it is an interesting problem to compute the shape of a net carrying a melon). However, all these problems are rather familiar to particle physicists since they arise in the study of ... Feynman diagrams. In particular, one may assign different Lagrangians to different links of the graph. In mechanical analogy this corresponds to the problem of the melon in the net with links of different tension and such a problem looks rather artificial and useless. However, it is what particle physicists study when particles have different masses and charges[3)].

The second confusion is the widely spread point of view that correlators in topological theories should be metric independent. Really, these are the simplest correlators and they appeared in the earliest examples. Mathematically speaking, such correlators are constant *functions* on the space of gravitational backgrounds. However, as it was realized later, general correlators should be considered not as functions but rather as the differential form on the space of gravitational backgrounds. It is not hard to guess that the condition of function being constant is replaced by the condition of differential form being annihilated by the action of $d = t^i \dfrac{\partial}{\partial t^i}$ (here t^i stand for coordinates on the space of gravitational backgrounds). Mathematicians call such differential forms closed.

The use of forms on graphs in 1-dimensional theory is well-known for many years. Parameters t^i are known as Schwinger parameters or proper times in the 1-dimensional theory on Feynman graphs. In order to get the contribution of such a graph in the target space amplitude or the correlator we should integrate differential forms over the space of Schwinger parameters. It is amazing that in the case of amplitudes in topological quantum mechanics we may promote top degree forms to forms of lower degree, and even to functions and study all this objects simultaneously. And here comes the prize – the tree level amplitudes satisfy quadratic relations known in mathematis as *homotopical algebra relations*! And relations including graphs with loops were not studied by mathematis – in this case physicist an may predict new mathematical objects and call it *quantum homotopical algebra structures* since loops come from quantum corrections!

In the most examples in quantum field theory relations among Feynman diagrams are just reflection of some symmetries of the action, and these relations become the Ward identities. It turns out that the same happens in quantum homological algebra. There is an action, called the Batalin–Vilkovisky (BV) action such that its exponent solve the famous master equation [10]. This may be considered as the set of symmetries of the action and leads to equations on amplitudes that may be also found as a relations on contributions of particular graphs.

2. Simplest properties of topological quantum mechanics. Quantum mechanics being the simplest quantum field theory can be described purely in terms of linear algebra: space of states V is a vector space with hermitian scalar product, and hamiltonian H is just a hermitian linear operator[4)].

Symmetries of the system correspond to operators, that commute with H. We will assume that theory contains odd or fermionic symmetry. This means that theory contains an operator of fermionic parity that physicist traditionally denote as $(-1)^F$ that squares to 1 and commutes with hamiltonian:

$$(-1)^F H = H(-1)^F; \quad ((-1)^F)^2 = 1. \tag{1}$$

States that are eigenfunctions of $(-1)^F$ with eigenvalues $+1$ (-1) are called bosonic (fermionic) states respectively and operator is called odd or fermionic if it changes the fermionic parity (turns bosonic states into fermionic and vice versa). In particular 1 means that hamiltonian itself is an even operator.

[3)] We cannot restrain from the following bothering remark: from this point of view the proper analogue of the graph is not a smooth worldsheet of a string but rather a piecewise smooth shape of a foam!

[4)] For the sake of simplicity we will ignore subtleties that can appear if vector space turns out infinite-dimensional.

Symmetry is called fermionic if its operator is odd. Such symmetry is traditionally denoted by Q, and we have:

$$QH = HQ, \;\; (-1)^F Q = -Q(-1)^F. \quad (2)$$

From this algebra it is clear that Q^2 is also an even symmetry, but we demand that it just equals to zero:

$$Q^2 = 0. \quad (3)$$

It mathematics this property means that Q is a differential.

In what follows we introduce the notion of a supercommutator $[,]_\pm$:

$$[A,B]_\pm = AB + (-1)^{\text{parity } A \cdot \text{parity } B} BA. \quad (4)$$

Note, that from equation (3) it follows that any operator of the form $[Q,A]_\pm$ supercommutes with Q. Such operators are called exact in mathematics.

Now we are ready to give the definition of the topological quantum mechanics: topological quantum mechanics is a quantum mechanics with an odd symmetry that squares to zero and such that its hamiltonian is exact:

$$H = [Q,G]_{pm} = QG + GQ. \quad (5)$$

The operator G is called superpartner of the hamiltonian. This operator is not uniquely defined, one can show that one can always take it to be odd with[5)] $G^2 = 0$.

The simplest example of topological quantum mechanics is the supersymmetric oscillator. Let a (a^+) be bosonic and ψ (ψ^+) fermionic annihilation (creation) operators. The hamiltonian of supersymmetric oscillator and operators Q and G are equal to:

$$H = \frac{\omega}{2}(a^+a + \psi^+\psi); \quad Q = \psi^+ a; \quad G = \frac{\omega}{2} a^+ \psi. \quad (6)$$

It is quite remarkable that such simple property (5) could cause interesting consequences.

The first consequence is that

$$I(T) = Tr(-1)^F \exp(-TH) \quad (7)$$

is T independent. Therefore $I(T) = I$ and I is called a Witten's index of the system.

[5)] Another name for topological quantum mechanics is $N = 1$ supersymmetric quantum mechanics. The latter is mostly known as a quantum mechanics with two symmetries Q_a with $Q_1^2 = Q_2^2 = 0$. However, after substitution $Q = Q_1 + iQ_2$ and $G = Q_1 - iQ_2$ we come back to (5).

The easiest way to see this is to consider the spectrum of the system. It consists of doublets at nonzero values of hamiltonian with opposite fermionic parity, $(a^+)^k|0\rangle$ and $(a^+)^{k-1}\psi^+|0\rangle$ (here $|0\rangle$ is the vacuum).The contribution of these doublets drops out of the expression for the index. Only vacuum contributes, and its contribution is equal to 1.

Since T could be considered as the volume of the 1-dimensional worldsheet, we may say that I is independent of the metric, and therefore depends only on the topology of the worldsheet, this explains the name of the theory.

Despite the simplicity of the reasoning above, it does not provide a conceptual explanation of metric independence of I. The real reason for metric independence is that derivative of $I(T)$ with respect to change of the metric (rescaling of T in our simplest example) is the correlator of H, and its exactness guarantees that this correlator equals to zero:

$$\frac{d}{dT} I(T) = -Tr(-1)^F \exp(-TH)H = \\ = -Tr(-1)^F \exp(-TH)(QG + GQ) = 0. \quad (8)$$

Really, if we move the Q in the second term all the way around under the trace we will get the first term but with the opposite sign!

Being inspired with such success with the index we may try to study correlators of other operators. The one point correlator of Φ (on Euclidean periodic worldsheet – circle S_1 of the length T) would look as follows:

$$\langle \Phi \rangle_{S_1;T} = Tr(-1)^F \exp(-TH)\Phi. \quad (9)$$

The attempt to prove its T independence (just like we did for an index) fails, really

$$\frac{d}{dT} \langle \Phi \rangle_{S_1;T} = Tr(-1)^F \exp(-TH)[Q,\Phi]_\pm G \quad (10)$$

and differs from zero for general Φ. However, Φ that has zero supercommutator with Q saves the game and its correlator is again T independent. Such Φ are called Q-closed and are also called zero-topological observables.

Reasoning in this way one can easily show that correlator of zero-topological observables on the circle

$$\langle \Phi_1(T_1) \ldots \Phi_k(T_k) \rangle_{S_1;T} = Tr(-1)^F \Phi_1 \times \\ \times \exp(-(T_1 - T_2)H)\Phi_2 \ldots \Phi_k \exp(-(T - T_1)H) \quad (11)$$

is independent of the distances (in time) between observables. Really, taking derivative with respect to T_1 replaces $\Phi_1(T_1)$ with $[H, \Phi_1(T_1)]_\pm$ and we get zero as above. Naively one may conclude that (11) is T_i independent – but it is not so. Correlator jumps when we are trying to interchange observables, therefore it depends only on their cyclic order. This is a one-dimensional version of famous link invariants that was interpreted by Witten as correlators of the Wilson line observables [3].

These properties of correlators on a circle can be generalized to transition amplitudes, i.e. correlators on an interval. The novelty here are boundary conditions that take the form of initial and final states:

$$\langle a|\Phi_1(T_1)\dots\Phi_k(T_k)|b\rangle = \\ = \langle a|\Phi_1 \exp(-(T_1-T_2)H)\Phi_2\dots\Phi_k|b\rangle. \qquad (12)$$

Such an amplitude depends only on the order of events happening at times $T_1,\dots,T_k$ if initial and final states are annihilated by Q:

$$\langle b|Q = Q|a\rangle = 0. \qquad (13)$$

Now, let us suppose that H has a nonnegative spectrum and that all vacua are annihilated not only by H but also by Q (like in the example of supersymmetric oscillator), then one can show that k-point correlators (12) can be rewritten in terms of 1-point correlator. Note, that these conditions are automatically satisfied if there is an hermitian metric on the space of states, such that $G = Q^+$ (that is how it happens in supersymmetric quantum mechanics, where operators Q_a are supposed to be real) and if the hamiltonian has a discrete spectrum.

Really, consider the two-point correlator and tend $T_1 - T_2$ to infinity. In this limit

$$\exp(-(T_1-T_2)H) \to \Pi = \sum_c |c\rangle\langle c| \qquad (14)$$

where $|c\rangle$ denote the basis in the space of vacua. Therefore,

$$\langle a|\Phi_1(T_1)\Phi_2(T_2)|b\rangle = \\ = \langle a|\Phi_1 \exp(-(T_1-T_2)H)\Phi_2|b\rangle \to \Pi = \\ = \sum_c \langle a|\Phi_1|c\rangle\langle c|\Phi_2|b\rangle. \qquad (15)$$

Moreover, even 1-point functions are not independent. Consider an algebra of zero observables:

$$\Phi_i\Phi_j = C_{ij}^k\Phi_k. \qquad (16)$$

In the limit $(T_1 - T_2) \to 0$ we get

$$\sum_c \langle a|\Phi_i|c\rangle\langle c|\Phi_j|b\rangle = C_{ij}^k\langle a|\Phi_k|b\rangle. \qquad (17)$$

This means that 1-point vacuum transition amplitudes form a representation of the algebra with structure constants C_{ij}^k.

Life becomes even more interesting when we include 1-observables in the game. We will do it in the next section.

3. Deformation theory and 1-observables. Not only generic physical system, but also system of very special type generally depend on some parameters[6]. Therefore, consider topological quantum mechanics that depends on parameters that we call t_i. In particular, consider equation $Q(t)^2 = 0$ and let us see what happens at the first order in t. We get:

$$\left[Q,\ \frac{\partial Q}{\partial t_i}\right]_\pm = 0. \qquad (18)$$

This equation means that $\partial Q/\partial t_i$ can be considered as a zero-observable Φ_i that we have just studied (note that the parity of t_i and Φ_i are opposite[7]).

Let us assume, for simplicity, that when we change parameter t_i the superpartner G is not changing. Therefore, we are going to study variation in t of correlators of zero-observables. In what follows we will use the remarkable integral representation for the variation of the evolution operator:

$$\frac{\partial e^{-TH}}{\partial t_i} = -\int_0^T dT_1 e^{-(T-T_1)H}\frac{\partial H}{\partial t_i}e^{-T_1H} = \\ = -\int_0^T dT_1 e^{-(T-T_1)H}[G,\Phi_i]_\pm e^{-T_1H}. \qquad (19)$$

Let us consider, for simplicity, variation of 1-point function on the circle:

$$\frac{\partial\langle\Phi_j\rangle_{S_1;T}}{\partial t_i} = \\ = \int_0^T dT_1 Tr(-1)^F e^{-(T-T_1)H}[G,\Phi_i]_\pm e^{-T_1H}\Phi_j = \\ = \int_0^T \langle\Phi_i^{(1)}(T_1)\Phi_j(0)\rangle_{S_1;T} \qquad (20)$$

where we have introduced the one-observable $\Phi_i^{(1)}$ that is a 1-differential on the worldsheet and that in Heisenberg representation takes the form:

$$\Phi_i^{(1)}(T_1) = [G,\Phi_i]_\pm(T_1)dT_1. \qquad (21)$$

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[6] Only perfect bodies look rigid but even they have parameters, like an overall scale.

[7] One may be surprised to see system with odd parameter t and even claim that such system is unphysical. However, the role of this parameter may be played either by some fermionic background field or by differential form. Here we use that odd parameters anticommute with any odd objects, with odd fields as well as with odd parameters.

Originally the notion of 1-observable (and higher observables in multidimensional theories) was introduced by Witten from the following considerations. He tried to construct nonlocal observable ω that would be integrated under the correlator and that would preserve the symmetry Q [4]. His idea was that if there is such observable Φ that solves what he called the topological descend equation

$$[Q,\omega(T_1)] = dT_1\frac{\partial}{\partial T_1}\Phi(T_1), \tag{22}$$

then under correlator

$$\int[Q,\omega(T_1)] = \int dT_1\frac{\partial}{\partial T_1}\Phi(T_1) = 0, \tag{23}$$

since the right hand side is the integral of the total derivative. However, this reasoning is only the first approximation to correct construction. Really, the expression above is not zero in general since, the integrand has well-known jumps due to supercommutators of Φ with other operators involved. Taking these supercommutators into account leads to the modified statement:

$$\begin{aligned}&\langle[Q,\omega(T_1)]\Phi_1\dots\Phi_k\rangle = \\ &= \langle[\Phi,\Phi_1]_\pm\dots\Phi_k\rangle + \dots\langle\Phi_1\dots[\Phi,\Phi_k]_\pm\rangle\end{aligned} \tag{24}$$

that is nothing but the Ward identities for the correlators $\langle\Phi_1\dots\Phi_k\rangle_t$ in the theory with the modified symmetry $Q+t\Phi$ in the first order in t – just as we discussed above. In order to completely demystify this issue we note that for Q-closed Φ the 1-observable ω that solves the descent equation (22) is just given up to the sign by the formula (21).

4. Problems in deformation theory. 4a. *Obstructions.* Interpretation of correlators of 1-observables as deformations of the theory is good as the first approximation, however, in has some problems.

The first problem is that deformed $Q(t) = Q + t_i\Phi_i$ generically does not square to zero at the second order in t. Really, for Q-closed Φ_i we have:

$$Q^2(t) = t_it_j[\Phi_i,\Phi_j]_\pm. \tag{25}$$

This would mean that $Q(t)$ is not a symmetry of the deformed hamiltonian $H(t) = Q(t)G + GQ(t)$. We may try to cure the problem by adding terms quadratic in t:

$$Q_{\mathrm{mod}(2)}(t) = Q + t_i\Phi_i + t_it_j\Phi_{ij}. \tag{26}$$

Therefore, the modified $Q_{\mathrm{mod}(2)}$ squares to

$$Q^2_{\mathrm{mod}(2)}(t) = t_it_j([Q,\Phi_{ij}]_\pm + [\Phi_i,\Phi_j]_\pm). \tag{27}$$

We would like to find such Φ_{ij} that it would take the righthand side of (27) to zero. However, it can not always be done. The obstruction to do this is hidden in the fact that while $[\Phi_i,\Phi_j]_\pm$ always (anti)commute with Q (i.e. it is closed), it is not (generically) an (anti)commutator, i.e. it is not exact. The space of closed operators moduli exact ones is called the space of cohomology of operators, and condition is that (anti)commutator of deforming operators has to vanish as an element of this space.

This situation is not so novel in quantum field theory. Let us recall gauge theory in lower dimension obtained from the gauge theory in higher dimensions by dimensionally reducing (or compactifying on a circle) of k dimensions. At the point with unbroken nonabelian symmetry we get k massless scalar fields ϕ_i in the adjoint representation. We know that massless fields are often the sign of the valley in the potential, and we may try to look here for such a valley. However, as we know, the (classical) potential is quartic

$$V = Tr([\phi_i,\phi_j])^2 \tag{28}$$

and naive valley is lifted in higher order in the deformation ϕ. Here the analogue of $Q(t)$ is the BRST operator in the ϕ background, and vanishing of potential means that obstruction is zero[8].

After improving $Q(t)$ at the second order we may meet problem in the third order in deformation parameter. Namely, in the third order in t we would need to solve:

$$t_it_jt_k[Q,\Phi_{ijk}]_\pm = -t_it_jt_k[\Phi_i,\Phi_{jk}]_\pm \tag{29}$$

and once again obstruction to solve this equation would be the image of the right hand side of (29) in cohomologies.

Problem of this kind are well-known in mathematics – they are called Torelli-like problems. In old days people studied situations when obstruction vanish (valleys of the potential, in physical language), now they are studying obstruction themselves (shape of the potential, in physical language).

4b. *Connection.* We have seen that 1-observables are related to deformations of the theory. Naively, one may think that correlator of one 1-observable and some number of zero-observables should be expressed as the first derivative of correlator with respect to parameters of the background. However, life is not so simple. Topological zero observables are different for different backgrounds

[8]However, we do not know how to make this analogy complete since we do not know what should correspond to correcting term Φ_{ij} in gauge theories.

(since they have to (anti)commute with $Q + t_i\Phi_i$, and latter depends on t). In order to take derivative we need to identify them somehow for different t. This problem is well-known in gauge theory, when one has to take derivatives of the charged field, and one has to pick up some identification of its phase at different points. In this case we say that we need connection.

When we look at correlator of one observables itself, we find the same problems. Recall, that performing a deformation of Q we had to solve equations $[Q, \Phi_{i_1\ldots i_m}]$ = *something*. Note, that we can always modify the solution by replacing

$$\Phi_{i_1\ldots i_m} \to \Phi_{i_1\ldots i_m} + V^i_{i_1\ldots i_m}\Phi_i. \tag{30}$$

One can check that this corresponds to local change of coordinates

$$t^i \to t^i + V^i_{i_1\ldots i_m} t^{i_1}\ldots t^{i_m}. \tag{31}$$

Therefore, in order to study the correlators of just one observable in the case when deformation is not obstructed we have to pick up special coordinate on the base of deformation, that is equivalent to take a flat torsion-free connection on the tangent space to the base of deformation.

However, if we pick (anti)commuting Φ_i there is an obvious solution to all deformation equations, the base of deformation gets equipped with the distinguished coordinate system and there is also a distinguished connection on the bundle of zero-observables.

5. Coupling to 1-dimensional topological gravity and quadratic relations among correlators. We have already observed appearance of gravity in topological theory in the form of the distance $T_i - T_{i+1}$ between the observables. However, as we will see in this section, in order to put together correlators of different observables and to see quadratic relations among them we need to study topological gravity.

We will promote bosonic distance T to a superdistance that is a pair (T, Ψ). In other terms, we will consider Ψ_i as dT_i and correlators (previously considered as functions of T_i) will become functions of T_i and dT_i, namely, they will become differential forms. The operator Q acting on matter field will be promoted to the total operator

$$Q^{(\mathrm{tot})} = Q + Q_{\mathrm{gravity}} = Q + \Psi_i\frac{\partial}{\partial T_i} = Q + d. \tag{32}$$

The evolution operator is promoted to the superevolution operator

$$U(T, dT) = \exp(-\{Q^{\mathrm{tot}}, TG\}) = \exp(-TH - dTG). \tag{33}$$

This presentation shows that the evolution operator depends on supermetric in the Q-exact manner, i.e. through the Q-exact term. In particular, it means, that U is Q^{tot} closed (i.e. that Q^{tot} is really an odd symmetry of the theory coupled to gravity):

$$\{Q + d, U(T, dT)\} = 0. \tag{34}$$

Therefore, the topological correlators that we have already studied are promoted to supertopological correlators. They are differential forms of indefinite degree. The degree zero part of supertopological correlator is just a topological correlator. The condition of T-independence of topological correlator is promoted to d-closeness of corresponding differential form. In order to save space we will write down explicitly supertopological correlator for 3 points on a line:

$$\begin{aligned}&\langle a|\Phi_1(T_1, dT_1)\Phi_2(T_2, dT_2)\Phi_3(T_3, dT_3)|b\rangle = \\ &= \langle a|\Phi_1 U(T_1 - T_2, d(T_1 - T_2)) \times \\ &\times \Phi_2 U(T_2 - T_3, d(T_2 - T_3))\Phi_3|b\rangle\end{aligned} \tag{35}$$

and

$$d\langle a|\Phi_1(T_1, dT_1)\Phi_2(T_2, dT_2)\Phi_3(T_3, dT_3)|b\rangle = 0. \tag{36}$$

It is clear that supertopological correlator is independent of the overall shifts in T_i, so it is a function on the coset M_3 (mathematicians call such coset moduli spaces), that is nothing else that R^2_+ and is parameterized by two positive coordinates $T_1 - T_2$ and $T_2 - T_3$. Moreover, it is clear that on M_3 the supertopological correlator is also closed.

This allows to write down relations on integrals supertopological correlators by integrating them along the boundary, that we will denote as follows:

$$A^a_{i_1,\ldots,i_m;b} = \int_{R_+^{(m-1)}}\langle a|\Phi_{i_1}(T_1, dT_1)\ldots\Phi_{i_m}(T_m, dT_m)|b\rangle. \tag{37}$$

Since supertopological correlators are closed, their integral along the boundary gives zero. So, the only thing is needed is to restrict supertopological correlator to the boundary. There are two types of boundaries, when coordinate (say, $T_1 - T_2$) goes to zero (ultraviolet) or to $+\infty$ (infrared). On the ultraviolet boundary operator is just replaced by a product of operators. On the infrared boundary evolution operator is replaced by projector to vacuum states. So we get:

$$A^a_{1,2;c}A^c_{3;b} + A^a_{1;c}A^c_{2,3,b} = f^i_{12}A^a_{i,3;b} + f^i_{23}A^a_{3,i;b}. \tag{38}$$

It is clear how to extend this to the case of an arbitrary number of points on the line.

After we get this remarkable quadratic equation (and its m-point generalizations known in mathematics as structure of A_∞-module of the algebra with structure constants f on the space of vacua), we may deduce some consequences. First of all, we may introduce generating parameters t_i and study

$$C^a_b(t) = A^a_{i_1,\dots,i_m;b} t_{i_1} \dots t_{i_m}. \tag{39}$$

From quadratic equations of the (38) we will see that it satisfies:

$$(D + C)^2 = 0 \tag{40}$$

where

$$D = c^k_{ij} t^i t^j \frac{\partial}{\partial t^k};\ c^k_{ij} = f^k_{ij} - f^k_{ji} \tag{41}$$

is what physicists used to call BRST operator for the Lie algebra with the structure constant c, and mathematicians call Chevalley differential. Equations (40) are known in mathematics as a structure of L_∞ module (for Lie algebra with the structure constants c) on the space of vacua (in physical literature this equation was first obtained in [12]).

Note, that due to symmetrization (due to permutation of t) one can interpret C^a_b as a generating function for correlator of many 1-observables in the presence of only one zero observable:

$$C^a_{i_1,\dots,i_m;b} = \langle a|\Phi^{(0)}_m \int_R \Phi^{(1)}_{i_1} \dots \int_R \Phi^{(1)}_{i_{m-1}} |b\rangle. \tag{42}$$

Therefore, we can conclude, that the proper treatment of correlator of higher observables as correlator in a system coupled to gravity allows to get interesting equations on them.

6. Topological theory on graphs and Feynman diagrams in BV theory. In the previous section we developed topological gravity approach to topological correlators. We may interpret line with points on it as a simplest tree with 2-valent vertices, one leaf and one root.

We will start our generalization with rooted trees with polyvalent vertices, these vertices have k inputs and one output, and may be considered as operators from $V^{\otimes k}$ to V. The simplest example is the 3-valent vertex that generates the bilinear operation. Surely, we will connect vertices by propagators U described above, so in order to preserve $Q^{\rm tot}$ symmetry we have to assume that Q commutes with vertex (considered as an operation); for 3-valent vertex this is nothing but the Leibnitz rule.

Like in the previous section, given a tree with K edges we may construct from vertices and propagators U a differential form on R^K_+, and this form would be closed. Integrating this form along the boundary we will get two type of contributions. Infrared contributions would correspond to decoupling of tree in two, and ultraviolet to shrinking of an edge and multiplication of K-valent and L-valent vertices into a $(K+L-1)$-valent vertex. Similarly, one can write down a quadratic relation, equating contribution of infrared and ultraviolet boundaries. This equation is known in mathematics as an equation of A_∞-morphism, and we are not going to present it here due to lack of space. We already presented an outline, so an interested reader can reconstruct either this equation or some of its particular example.

However, why should we restrict our attention only to rooted trees? We may wish to consider any graph. However, graphs could be oriented or not oriented[9].

For the oriented graph we need to introduce vertices with several inputs and outputs (in mathematics they are called PROP's). Then we construct differential form as above, integrate it along the boundary and observe that now we have a new infrared component corresponding to lowering the number of the loops in the graph by one.

For nonoriented graph we may introduce the bilinear pairing on the space V that is preserved by Q, G and such that it makes K-valent vertices into a K-valent cyclically symmetric tensors.

Therefore, just considering graphs with the real numbers on edges we can construct many operations that generalize in a highly nontrivial way ∞-structures, that were introduced by mathematicians.

Note, that just this would be a great success. However, physics can go further, and interpret the graphs introduced above as Feynman graphs in the theory of Batalin–Vilkovisky integrals [15].

We will start with the case of rooted oriented trees. In this case we may consider $V \oplus V^*[1]$ as a space of fields in the theory. Here is just a Hilbert space of our quantum mechanics, V^* is the dual space, and [1] is a fancy way to say that we are inverting statistic (dual to bosonic field is fermionic and vice versa). Let Z^A be coordinates in V and P_A in $V^*[1]$. We will introduce an action:

$$S_{BV}(P, Z) = P_A W^A(Z) \tag{43}$$

where

$$W^A(Z) = Q^A_B Z^B + t_i \Phi^A_{i,B} + F^A_{BC} Z^B Z^C + \dots, \tag{44}$$

Q^A_B and $\Phi^A_{i,B}$ are matrix elements of operators Q and Φ_i.

[9] Just like trees. There are rooted trees with orientation on their edges from leaves to the root, and abstract trees – the latter are not oriented.

Then we decompose spaces V and $V^*[1]$ into infrared and ultraviolet parts. Infrared parts would consist of vacua, and ultraviolet part would be the rest.

We will integrate $\exp S/\hbar$ along the submanifold of ultraviolet variables:

$$G_B^A Z^B = 0;\ P_B G_A^B = 0.$$

The resulting induced action is $S_{\text{ind}}(P_{IR}, Z_{IR}, \hbar)$ that is a sum over graphs with h being a genus counting parameter.

Due to the well-known theorem of BV integral [16]:

$$\frac{\partial}{\partial_{P_{IR,A}}} \frac{\partial}{\partial_{Z_{IR}^A}} \exp(S_{ind}(P_{IR}, Z_{IR}, \hbar)/\hbar) = 0$$

and this is the physical meaning of the quantum generalizations of the infinity structures.

7. Concluding remarks. It is clear that what we have covered here is just an introduction in the rapidly developing subject of topological theories. In particular, just by replacing intervals by cylinders one can get many new structures such as Commutativity equations and their relations to WDVV equations [14]. The string-motivated generalization of graphs with numbers on edges to moduli space of Riemann surface leads to Zwiebach invariants introduced in [17].

However, this subject still contains a lot of open questions. We will mention some of them.

The Donaldson theory is still waiting for the proper treatment including four dimensional version of topological gravity.

Relation between Virasoro constraints in topological strings and BV master equations is not clear at all.

It is even not clear that topological theories have no divergences.

And the most intriguing question is whether the four dimensional gravity theory in the Palatini formulation (that is also a topological theory) is divergent!

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1. E. Witten, Nucl. Phys. B **188**, 513 (1981).
2. E. Witten, Nucl. Phys. B **340**, 281 (1990).
3. E. Witten, Commun. Math. Phys. **121**, 351 (1989).
4. E. Witten, Commun. Math. Phys. **117**, 353 (1988).
5. V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **223**, 445 (1983).
6. N. Berkovits, JHEP **0009**, 046 (2000).
7. N. Berkovits, JHEP **0209**, 051 (2002).
8. R. Dijkgraaf, S. Gukov, A. Neitzke, and C. Vafa, hep-th/0411073.
9. N. Nekrasov, hep-th/0412021.
10. I. A. Batalin, and G. A. Vilkovisky, Phys. Lett. B **102**, 27 (1981); Phys. Rev. D **28**, 2567 (1983).
11. E. Witten, Commun. Math. Phys. **118**, 411 (1988).
12. V. Lysov, Pisma v Zh.Eksp.Teor.Fiz. **76**, 855 (2002).
13. M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Commun. Math. Phys. **165**, 311 (1994).
14. A. Losev and I. Polyubin, Pis'ma v ZhETF **77**, 59 (2003).
15. M. Movshev and A. Schwarz, Nucl. Phys. B **681**, 324 (2004).
16. A. Schwarz, Commun. Math. Phys. **158**, 373 (1993).
17. A. S. Losev and S. V. Shadrin, math.QA/0506039.