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Correlation functions of descendants in the scaling Lee–Yang model

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Correlation functions of the composite field $T\bar{T}$ in the scaling Lee–Yang model are studied. Using the analytic expression for form factors of this operator recently proposed by G. Delfino and G. Niccoli, hep-th/0407142 [1], we show numerically that the constraints on the $T\bar{T}$ expectation values obtained in A. B. Zamolodchikov, hep-th/0401146 [2] and the additional requirement of asymptotic behavior lead to a perfect agreement with the ultraviolet asymptotic predicted by the conformal perturbation theory.

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In the present work, we use two different approaches to massive integrable quantum field theories. In the first approach, the massive theory is considered as a perturbation of a certain conformal field theory (being a fixed point of the renormalization group flow) by a relevant operator [3]. The structure of the space of local operators does not change along the renormalization group flow; therefore, the space of local operators of the massive theory is assumed to be isomorphic to the corresponding conformal field theory space. This space consists of primary operators and descendants [4]. The correlation functions are calculated using operator product expansions

$$\langle A_i(x)\, A_j(0)\rangle = \sum_k C_{ij}^k(x,g)\,\langle A_k\rangle\,, \tag{1}$$

where g is the perturbing coupling constant. The structure functions of the operator algebra $C_{ij}^k(x,g)$, are assumed to be analytic in g, provided that the renormalized fields A_i are chosen to have definite dimensions. This hypothesis provides the way to expand the structure functions in a perturbation series in the coupling constant. In the zeroth order, the $C_{ij}^k(x,g)$ coincide with the structure functions of conformal field theory. This procedure, called Conformal Perturbation Theory, was invented in [5]. The vacuum expectation values $\langle A_i\rangle$ depend on g nonanalytically and cannot be calculated using perturbation theory. It follows from dimensional analysis that

$$\langle A_i\rangle = Q_i g^{\frac{\Delta_i}{1-\Delta}},$$

where Q_i is independent of g, Δ_i is the dimension of the field A_i, and Δ is the dimension of the perturbation. The vacuum expectation values of primary fields and the first nontrivial descendants were obtained in [6–8]. Taking into account that $\Delta < 1$ and the dimensions Δ_i are increasing, expression (1) represents the series in increasing powers of x, and we can restrict ourself to a small number of terms for $x << 1$.

In the second (form-factor) approach, matrix elements of local operators in the basis of asymptotic states are determined from the Smirnov axioms [9], given the S-matrix and mass spectrum. Correlation functions then can be represented as a spectral decomposition.

We consider the scaling Lee–Yang model

$$S = S_{M(2/5)} + g\int \varphi\, \mathrm{d}^2x, \tag{2}$$

i.e., the $M(2/5)$ minimal model of conformal field theory perturbed by the field $\varphi = \phi_{1,3}$, which is the only nontrivial primary field in it. The model $M(2/5)$ has the central charge $c = -22/5$. The space of fields in this model consists of two primary fields: the identity operator $I = \phi_{1,1} = \phi_{1,4}$ and the field $\varphi = \phi_{1,2} = \phi_{1,3}$ with the right and left dimensions $\Delta = \bar{\Delta} = -1/5$ and their descendants. It is convenient to consider the trace of the stress tensor Θ,

$$\Theta(x) = \pi g\,(1-\Delta)\,\varphi(x)\,. \tag{3}$$

The vacuum expectation value of $\langle\Theta\rangle$ was obtained in [10]

$$\langle\Theta\rangle = -\frac{\pi m^2}{4\sqrt{3}}, \tag{4}$$

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where m is the particle mass. In what follows, we also use the relation

$$g = i\frac{2^{\frac{1}{5}}5^{\frac{3}{4}}}{16\pi^{\frac{6}{5}}}\frac{\left(\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\right)^{\frac{12}{5}}}{\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{4}{5}\right)}m^{\frac{12}{5}} \quad (5)$$

between the coupling g and the scale m of the theory, found in [10, 11].

The form factors of the operator $\langle\Theta\rangle$ were found in [5]. The form factors of the operator $T\overline{T} = L_{-2}\overline{L}_{-2}I$ in this model were recently obtained in [1]. We use this expression to numerically calculate the correlation functions $G(m|x|) = m^{-6}\langle T\overline{T}(x)\,\Theta(0)\rangle$ and $H(m|x|) = m^{-8}\langle T\overline{T}(x)\,T\overline{T}(0)\rangle$ up to three-particle terms in spectral expansions, and we compare them with the leading terms in conformal perturbation theory (1).

The first three nonzero vacuum expectation values in expansion (1) are $\langle\varphi\rangle$, $\langle I\rangle = 1$, and $\langle L_{-2}\overline{L}_{-2}I\rangle = \langle T\overline{T}\rangle$. We recall the operator product expansions of the stress tensor in a conformal field theory [4]:

$$T(z)\,\phi(0) = \frac{\Delta}{z^2}\phi(0) + \frac{1}{z}\partial\phi(0) + \dots, \quad (6)$$

$$T(z)\,T(0) = \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \dots. \quad (7)$$

This gives

$$C^{I}_{T\overline{T},\varphi}(x) = 0, \quad C^{\varphi}_{T\overline{T},\varphi}(x) = \Delta^2|x|^{-4},$$

$$C^{I}_{T\bar{T},T\bar{T}}(x) = \left(\frac{c}{2}\right)^2|x|^{-8}, \quad C^{\varphi}_{T\bar{T},T\bar{T}}(x) = 0.$$

We here skip the details of the first-order calculations for the structure functions. All nessesary formulas can be found in [5, 12]. Using (3) and (4) we finally obtain the expression for $\langle T\overline{T}(x)\,\Theta(0)\rangle$:

$$G_{\text{UV}}(m|x|) = -\frac{\pi}{100\sqrt{3}}(m|x|)^{-4} + g_1(2\ln\mu|x| + g_2)(m|x|)^{-\frac{8}{5}} + g_3(2\ln\mu|x| + g_4)(m|x|)^{-\frac{6}{5}}, \quad (8)$$

$$g_1 = -\frac{3^{\frac{3}{2}}\pi^{\frac{4}{5}}\left(\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\right)^{\frac{12}{5}}\left(\Gamma\left(\frac{1}{5}\right)\right)^{\frac{3}{2}}\left(\Gamma\left(\frac{2}{5}\right)\right)^{\frac{1}{2}}}{2^{\frac{19}{5}}5^{\frac{17}{4}}\left(\Gamma\left(\frac{3}{5}\right)\right)^{\frac{3}{2}}\left(\Gamma\left(\frac{4}{5}\right)\right)^{\frac{5}{2}}},$$

$$g_2 = 2\psi(2) - \psi\left(-\frac{1}{5}\right) - \psi\left(\frac{1}{5}\right) - \frac{115}{18},$$

$$g_3 = \frac{27}{\pi^{\frac{2}{5}}2^{\frac{13}{5}}5^{\frac{7}{2}}}\frac{\left(\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\right)^{\frac{24}{5}}}{\left(\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{4}{5}\right)\right)^2},$$

$$g_4 = 2\psi(2) - \psi\left(-\frac{2}{5}\right) - \psi\left(\frac{2}{5}\right) - \frac{55}{14}.$$

In the same way for $\langle T\overline{T}(x)\,T\overline{T}(0)\rangle$, we have

$$H_{\text{UV}}(m|x|) = \left(\frac{c}{2}\right)^2(m|x|)^{-8} + \frac{\pi\Delta^2(1-\Delta)}{\sqrt{3}} \times \left(4\ln\mu|x| + \frac{c-4\Delta^2}{2\Delta(\Delta-1)}\right)(m|x|)^{-6}. \quad (9)$$

In the first-order calculations, we have faced the resonance problem [5], leading to the undefined coefficient μ in the subleading terms in (8) and (9).

The mass spectrum of the scaling Lee–Yang model consists of one particle A. Correlation functions can be expressed through the form factors of local operators as spectral sums. For example, the two-point Euclidean correlation function of the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ has the form

$$\langle\mathcal{O}_1(x)\,\mathcal{O}_2(0)\rangle = \sum_{n=0}^{\infty}\int\frac{d\theta_1\dots d\theta_n}{n!\,(2\pi)^n}F_n^{\mathcal{O}_1}(\theta_1,\dots\theta_n) \times F_n^{\mathcal{O}_2}(\theta_n,\dots\theta_1)\,e^{-m|x|\sum\limits_{i=1}^{n}\cosh\theta_i}. \quad (10)$$

The expressions for the first four form factors of the operator Θ have the forms [5]:

$$F_0^{\Theta} = -\frac{\pi m^2}{4\sqrt{3}}, \quad (11)$$

$$F_1^{\Theta} = -\frac{i\pi m^2}{2^{\frac{5}{2}}3^{\frac{1}{4}}v(0)}, \quad (12)$$

$$F_2^{\Theta}(\theta_1,\theta_2) = \frac{\pi m^2}{2}\frac{f(\theta_1-\theta_2)}{4v^2(0)}, \quad (13)$$

$$F_3^{\Theta}(\theta_1,\theta_2,\theta_3) = \frac{i3^{\frac{1}{4}}\pi m^2}{2^{\frac{7}{2}}v^3(0)}\prod_{i<j}^{3}f(\theta_i-\theta_j)\left(1 + \frac{1}{8\prod\limits_{i<j}\cosh\frac{\theta_i-\theta_j}{2}}\right), \quad (14)$$

where

$$f(\theta) = \frac{\cosh\theta - 1}{\cosh\theta + 1/2}v(i\pi-\theta)\,v(-i\pi+\theta), \quad (15)$$

$$v(\theta) = \exp\left(2\int\limits_0^{\infty}\frac{\sinh\frac{t}{2}\sinh\frac{t}{3}\sinh\frac{t}{6}}{t\sinh^2 t}e^{\frac{i\theta t}{\pi}}dt\right). \quad (16)$$

The following expression for the form factors of the operator $T\overline{T}$ was obtained in [1] using the restriction on the growth at infinity, the asymptotic factorization properties, and the relation for the expectation value of $T\overline{T}$ obtained in [2]:

$$F_n^{T\overline{T}} = m^2 \left(a \left(\sigma_1^{(n)} \bar{\sigma}_1^{(n)} \right)^2 + c\sigma_1^{(n)} \bar{\sigma}_1^{(n)} + d \right) F_n^{\Theta} +$$

$$+ bF_n^{\mathcal{K}_3} + em^4 \delta_{n,0}, \qquad (17)$$

where

$$\sigma_1^{(n)} = \sum_{i=1}^{n} x_i, \quad \bar{\sigma}_1^{(n)} = \sum_{i=1}^{n} \frac{1}{x_i}. \qquad (18)$$

For $n < 3$, $F_n^{\mathcal{K}_3}$ is the solution equal to zero, and at $n = 3$,

$$F_n^{\mathcal{K}_3} (\theta_1, \theta_2, \theta_3) = -i \left(\frac{3}{4} \right)^{\frac{3}{4}} \frac{m^2}{v^3 (0)} \times$$

$$\times \prod_{i<j}^{3} f (\theta_i - \theta_j) \left(\cosh (\theta_i - \theta_j) + \frac{1}{2} \right). \qquad (19)$$

The constants a, b, d, are e are

$$a = \frac{\langle \Theta \rangle}{m^2}, \quad b = -\frac{\langle \Theta \rangle^2}{m^4}, \qquad (20)$$

$$d = -\frac{2}{m^2} \langle \Theta \rangle, \quad e = -\frac{\langle \Theta \rangle^2}{m^4}, \qquad (21)$$

where

$$\langle \Theta \rangle = -\frac{\pi m^2}{4\sqrt{3}}. \qquad (22)$$

The constant c is not determined, which corresponds to an ambiguity $T\overline{T} \to T\overline{T} + \#\partial\bar{\partial}\varphi$ in the definition of the operator $T\overline{T}$ outside the critical point, because the dimensions of operators $T\overline{T}$ and $\partial\bar{\partial}\varphi$ satisfy the resonance condition [5]. The coefficient a is determined only from the restriction on the growth at infinity and the asymptotic factorization condition, and the coefficients d and e, from the growth restriction and Zamolodchikov relation for the stress-tensor expectation value. The coefficient b is determined from each of these sets of conditions independently.

Formulas (11)–(14), (17), and (19) lead to the spectral expansion of the correlation functions $G (m |x|)$ and $H (m |x|)$ up to three-particle terms:

$$G_{\mathrm{IR}} (m |x|) = \left(\frac{\pi}{4\sqrt{3}} \right)^3 +$$

$$+ G_1 (m |x|) + G_2 (m |x|) + G_3 (m |x|) + \ldots, \qquad (23)$$

$$H_{\mathrm{IR}} (m |x|) = \left(\frac{\pi}{4\sqrt{3}} \right)^4 +$$

$$+ H_1 (m |x|) + H_2 (m |x|) + H_3 (m |x|) + \ldots, \qquad (24)$$

where

$$G_1 (x) = -\frac{\pi}{32\sqrt{3}v^2 (0)} (a + c + d) K_0 (x), \qquad (25)$$

$$G_2 (x) = \frac{1}{128v^4 (0)} \int_0^\infty \Big(4a (1 + \cosh\theta)^2 + 2c (1+ \cosh\theta) + d \Big) g (\theta) K_0 \left(2x \cosh \frac{\theta}{2} \right) d\theta, \qquad (26)$$

$$G_3 (x) = \frac{1}{32\pi^2 v^6 (0)} \int_0^\infty \int_0^\infty \Big(-\frac{\pi}{32\sqrt{3}} B (\theta, \chi) \times \left(a A^4 (\theta, \chi) + c A^2 (\theta, \chi) + d \right) + b C (\theta, \chi) \Big) B (\theta, \chi) \times$$

$$\times g (\theta) g (\chi) g (\theta - \chi) K_0 (A (\theta, \chi) x) d\theta d\chi, \qquad (27)$$

$$H_1 (x) = -\frac{\pi}{32\sqrt{3}v^2 (0)} (a + c + d)^2 K_0 (x), \qquad (28)$$

$$H_2 (x) = \frac{1}{128v^4 (0)} \int_0^\infty \Big(4a (1 + \cosh\theta)^2 + 2c (1+ \cosh\theta) + d \Big)^2 g (\theta) K_0 \left(2x \cosh \frac{\theta}{2} \right) d\theta, \qquad (29)$$

$$H_3 (x) = -\frac{\sqrt{3}}{\pi^3 v^6 (0)} \int_0^\infty \int_0^\infty \Big(-\frac{\pi}{32\sqrt{3}} B (\theta, \chi) \times \left(aA^4 (\theta, \chi) + cA^2 (\theta, \chi) + d \right) + b C (\theta, \chi) \Big)^2 \times$$

$$\times g (\theta) g (\chi) g (\theta - \chi) K_0 (A (\theta, \chi) x) d\theta d\chi, \qquad (30)$$

$$g (\theta) = f (\theta) f (-\theta), \qquad (31)$$

$$A (\theta, \chi) = \sqrt{3 + 2 (\cosh\theta + \cosh\chi + \cosh (\theta - \chi))},$$

$$B (\theta, \chi) = 1 + \frac{1}{8 \cosh \frac{\theta}{2} \cosh \frac{\chi}{2} \cosh \frac{\theta-\chi}{2}},$$

$$C (\theta, \chi) = \left(\cosh\theta + \frac{1}{2} \right) \left(\cosh\chi + \frac{1}{2} \right) \times \left(\cosh (\theta - \chi) + \frac{1}{2} \right).$$

Table 1

Numerical data for the correlation function $\langle T\bar{T}(x)\Theta(0)\rangle$

$m\,\lvert x\rvert$	G_{IR} up to 2 particles	G_{IR} up to 3 particles	G_{UV} leading term	G_{UV} first order
0.00001	−1.85653286e+18	−1.81300206e+18	−1.81380007e+18	−1.81380007e+18
0.00010	−1.85653275e+14	−1.81300214e+14	−1.81380007e+14	−1.81380007e+14
0.00100	−1.85652408e+10	−1.81300401e+10	−1.81380007e+10	−1.81379845e+10
0.00200	−1.16031262e+09	−1.13312746e+09	−1.13362505e+09	−1.13362023e+09
0.00400	−7.25161988e+07	−7.08199961e+07	−7.08515654e+07	−7.08501625e+07
0.00600	−1.43231670e+07	−1.39888795e+07	−1.39953709e+07	−1.39946955e+07
0.00800	−4.53151109e+06	−4.42603060e+06	−4.42822284e+06	−4.42782279e+06
0.01000	−1.85589100e+06	−1.81282105e+06	−1.81380007e+06	−1.81353444e+06
0.02000	−1.15890841e+05	−1.13251558e+05	−1.13362505e+05	−1.13289636e+05
0.03000	−2.28615882e+04	−2.23527759e+04	−2.23925935e+04	−2.23590156e+04
0.04000	−7.22104580e+03	−7.06438900e+03	−7.08515654e+03	−7.06596919e+03
0.05000	−2.95154366e+03	−2.88923575e+03	−2.90208012e+03	−2.88972936e+03
0.06000	−1.41992160e+03	−1.39078944e+03	−1.39953709e+03	−1.39095957e+03
0.07000	−7.64321784e+02	−7.49094702e+02	−7.55435266e+02	−7.49155160e+02
0.08000	−4.46655159e+02	−4.38020065e+02	−4.42822284e+02	−4.38041748e+02
0.10000	−1.81664934e+02	−1.78361775e+02	−1.81380007e+02	−1.78368395e+02
0.15000	−3.50788352e+01	−3.45331307e+01	−3.58281496e+01	−3.45550348e+01
0.20000	−1.07663389e+01	−1.06227468e+01	−1.13362505e+01	−1.06572736e+01

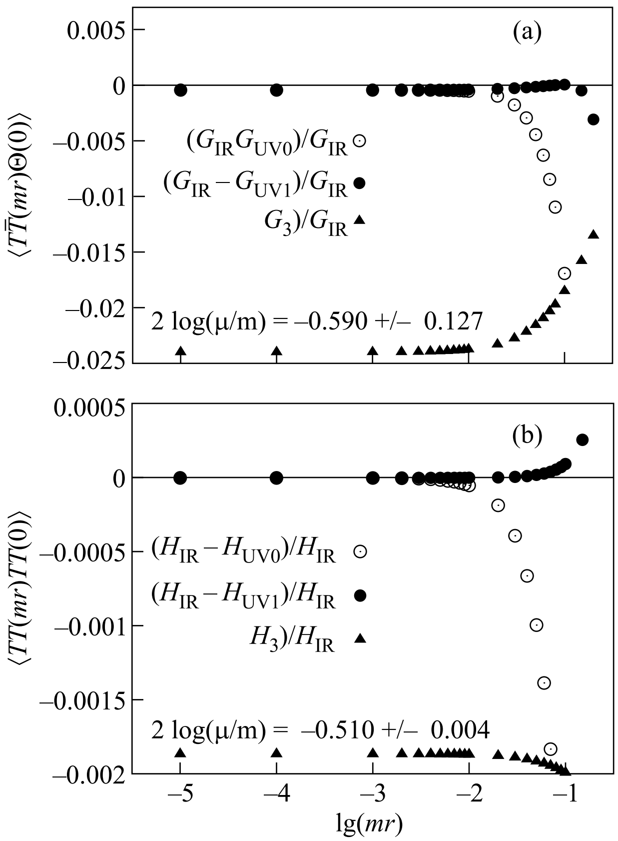
Table 2

Numerical data for the correlation function $\langle T\bar{T}(x)T\bar{T}(0)\rangle$

$m\,\lvert x\rvert$	H_{IR} up to 2 particles	H_{IR} up to 3 particles	H_{UV} leading term	H_{UV} first order
0.00001	4.84902564e+40	4.83998645e+40	4.84000118e+40	4.84000118e+40
0.00010	4.84902561e+32	4.83998641e+32	4.84000118e+32	4.84000114e+32
0.00100	4.84901983e+24	4.83998034e+24	4.83999827e+24	4.83999495e+24
0.00200	1.89414518e+22	1.89061384e+22	1.89062440e+22	1.89061959e+22
0.00400	7.39895858e+19	7.38516043e+19	7.38525142e+19	7.38518211e+19
0.00600	2.88693111e+18	2.88154509e+18	2.88161097e+18	2.88155315e+18
0.00800	2.89015215e+17	2.88475727e+17	2.88486384e+17	2.88476475e+17
0.01000	4.84879446e+16	4.83973781e+16	4.84000106e+16	4.83974909e+16
0.02000	1.89382175e+14	1.89026924e+14	1.89062538e+14	1.89026936e+14
0.03000	7.38795913e+12	7.37402153e+12	7.37692595e+12	7.37399344e+12
0.04000	7.39439146e+11	7.38034934e+11	7.38525556e+11	7.38027952e+11
0.05000	1.24017793e+11	1.23780576e+11	1.23903991e+11	1.23778474e+11
0.06000	2.88317563e+10	2.87761876e+10	2.88161166e+10	2.87754224e+10
0.07000	8.39672143e+09	8.38041127e+09	8.39577991e+09	8.38008907e+09
0.08000	2.88379417e+09	2.87814821e+09	2.88486545e+09	2.87799616e+09
0.09000	1.12334075e+09	1.12112407e+09	1.12435943e+09	1.12104556e+09
0.10000	4.83278636e+08	4.82317554e+08	4.83999982e+08	4.82273987e+08
0.15000	1.87883587e+07	1.87496455e+07	1.88849209e+07	1.87448871e+07
0.20000	1.87207209e+06	1.86810743e+06	1.89062486e+06	1.86704234e+06

As mentioned above, the ambiguity in the definition of the operator $T\overline{T}$ does not affect the leading UV order. Nevertheless, it is interesting to establish the exact correspondence between the coefficients μ and c. It happens that the precision of the form-factor calculations is insufficient for this purpose.

In Figure, we show the results of fitting IR data obtained for $c = 0$ by the UV expansions. In view of the rapid increase of the correlation functions considered, we use a logarithmic scale and also the ratios of the corresponding contributions for better visibility. The ratios of the three-particle contributions G_3/G_{IR} and H_3/H_{IR}



The results of fitting the IR data for $c = 0$ by the UV expansions with the fitting parameter μ

(triangles) are given in the figure to visibly demonstrate the degree of agreement between the IR and UV data and also to estimate in which interval we expect the IR data to be valid. As a result of such an estimation, we fitted on the interval [0.01, 0.2]. Further, we show the degree of agreement of the IR data with the zeroth-order (open circles) and with the first-order (filled circles) UV expansions. The errors in determining the fitting parameters shows that we must take the higher-order form-factor contributions into account.

A comparison of the numerical values of UV expansions (8) and (9) with form-factor expansions (23) and (24) up to three-particle contributions for $10^{-5} < m|x| < 0.2$ is shown in Tables 1 and 2. In the IR expansion, we use $c = 0$. The results of the first-order UV calculations are given for the best-fit value of $2\log(\mu/m) = -0.51$. It can be seen that the UV and IR expansions coincide with sufficiently good accuracy.

In conclusion, we emphasize that this comparison confirms the construction for the form factors of operator $T\overline{T}$ proposed in [1]. Indeed, we note that the two-particle terms G_2 and H_2 are highly sensitive to the value of the parameter a, and the three-particle terms G_3 and H_3, to the value of the parameter b (because the corresponding terms in the integrands in (26), (27), (29), and (30) have the greatest increase at infinity) and are weakly sensitive to the values of the other parameters. Therefore, the fact that the sums of the first three terms of the form-factor expansions (i.e., zero-, one-, and two-particle) coincide with the UV expansion with an accuracy up to three digits confirms the value of the parameter a and the assumption of asymptotic behavior for descendent operators in the scaling Lee–Yang model [1]. Including three-particle terms improves the convergence up to five digits which confirms the value of the parameter b and all conjectures used to determine it [1, 2].

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1. G. Delfino and G. Niccoli, hep-th/0407142, Nucl. Phys. B **707**, 381 (2005).
2. A. B. Zamolodchikov, hep-th/0401146 (2004).
3. A. B. Zamolodchikov, Adv. Stud. in Pure Math. **19**, 641 (1989).
4. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
5. Al. B. Zamolodchikov, Nucl. Phys. B **348**, 619, (1990).
6. S. Lukyanov and A. Zamolodchikov, Nucl. Phys. B **493**, 571 (1997); hep-th/9611238.
7. V. Fateev, D. Fradkin, S. Lukyanov et al., Nucl. Phys. B **540**, 587 (1999); hep-th/9807236.
8. V. Fateev, S. Lukyanov, A. Zamolodchikov et al., Nucl. Phys. B **516**, 652 (1998) hep-th/9709034.
9. F. Smirnov, *Formfactors in completely integrable models of quantum field theory*, World Scientific, 1992.
10. Al. B. Zamolodchikov, Nucl. Phys. B **342**, 695 (1990).
11. Al. B. Zamolodchikov, Int. J. Mod. Phys., A **10**, 1125 (1995).
12. A. A. Belavin, V. A. Belavin, A. V. Litvinov et al., hep-th/0309137; Nucl. Phys. B **676**, (2004).