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Bifurcations and stability of internal solitary waves

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We study both supercritical and subcritical bifurcations of internal solitary waves propagating along the interface between two deep ideal fluids. We derive a generalized nonlinear Schrödinger equation to describe solitons near the critical density ratio corresponding to transition from subcritical to supercritical bifurcation. This equation takes into account gradient terms for the four-wave interactions (the so-called Lifshitz term and a nonlocal term analogous to that first found by Dysthe for pure gravity waves) as well as the six-wave nonlinear interaction term. Within this model we find two branches of solitons and analyze their Lyapunov stability.

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1. Bifurcations of solitons were first observed for gravity-capillary waves for the deep water case in the numerical simulations of Longuet- Higgins [1] and explained later in [2 – 6]. In the shallow water case gravity-capillary solitons also undergo bifurcations. It happens at a small vicinity of the critical Bond number $Bo = (l_c/h)^2 = 1/3$ where l_c is the capillary length and h the depth (for details see [7]). Far from $Bo = 1/3$ solitons represent elevation or depression of the fluid surface as it was confirmed in the recent experiments [8].

Such bifurcations have a lot of common features with phase transitions. The analog to the phase transition of the second kind is a supercritical bifurcation where the soliton amplitude vanishes smoothly as the critical velocity is approached. Such a bifurcation occurs due to the Cherenkov resonant interaction of solitons with linear dispersive waves. Such a resonance occurs when the soliton velocity coincides with the minimum (or maximum) phase velocity of linear waves ($\equiv$ critical soliton velocity V_{cr}). Therefore, as their velocity approaches V_{cr}, solitons are transformed into envelope solitons: their form is universal and obtained from the stationary nonlinear Schrödinger equation (NLSE). The soliton amplitude vanishes as $(V - V_{cr})^{1/2}$, which is typical for second kind phase transitions. For subcritical bifurcations, which are the analog of phase transitions of the first kind, solitons at the bifurcation point undergo a jump in amplitude. If this jump is small enough, then the subcritical bifurcation is close to the supercritical one. In this case, a perturbation technique can be developed along the same lines as, for instance, near tri-critical points. The transition from the supercritical bifurcation to the subcritical one occurs when the four-wave coupling coefficient changes its sign (the nonlinearity changes from focusing to defocusing). As shown by Dias and Iooss [9], such a transition occurs for internal solitons propagating along the interface between two fluids when their density ratio $\rho = \rho_1/\rho_2 < 1$ takes the critical value $\rho_{cr} = (21 - 8\sqrt{5})/11 \approx 0.283$. In order to describe the soliton shape near ρ_{cr} one needs to take into account next order terms beyond the classical cubic NLS equation. As shown in this paper, these nonlinearities come from taking into account gradient terms for the four-wave interactions, including a local term relative to the soliton amplitude and its first spatial derivative (the so-called Lifshitz term [10, 11]) and a nonlocal term which has a structure similar to that for gravity waves found by Dysthe [12], and also the six-wave interaction term. In this paper we demonstrate how in the framework of the resulting generalized NLS equation it is possible to investigate both kinds of bifurcations of interfacial solitons as well as their stability. The stability analysis is based on the proof of boundedness of the Hamiltonian by means of the integral embedding inequalities of Sobolev type.

2. Consider two ideal fluids with different densities $\rho_{1,2}$ in the presence of gravity with acceleration g acting down the vertical $z-$axis and capillarity with interfacial tension σ. The light fluid, labelled 1, occupies the region $\infty > z > \eta(x,t)$, and the heavy one, labelled 2, occupies the region $-\infty < z < \eta(x,t)$. Flows of both fluids are assumed to be potential and two-dimensional:

$$\mathbf{v}_{1,2} = \nabla\phi_{1,2},$$

where the velocity potentials $\phi_{1,2}$ satisfy the Laplace equation

$$\Delta\phi_{1,2} = 0, \qquad (1)$$

with $\phi_{1,2} \to 0$ as $|z| \to \infty$. On the interface $z = \eta(x,t)$, there are two (kinematic and dynamic) boundary conditions:

$$\frac{\partial \eta}{\partial t} = \left(-\eta_x \frac{\partial \phi_{1,2}}{\partial x} + \frac{\partial \phi_{1,2}}{\partial z}\right)_{z=\eta} \equiv U, \quad (2)$$

$$\rho_2 \left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}(\nabla \phi_2)^2 + g\eta\right) - - \rho_1 \left(\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\nabla \phi_1)^2 + g\eta\right) = \sigma \frac{\partial}{\partial x}\left(\frac{\eta_x}{\sqrt{\eta_x^2+1}}\right). \quad (3)$$

As was shown in [16], equations (2), (3) subject to (1) can be represented in Hamiltonian form for $\Psi = (\phi_2 - \rho\phi_1)|_{z=\eta}$ and η :

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad (4)$$

where the Hamiltonian coincides with the total energy:

$$H = \frac{1}{2}\int \left[U\Psi + (1-\rho)g\eta^2 + 2\sigma\left(\sqrt{1+\eta_x^2} - 1\right)\right] dx.$$

Here the first term is the kinetic energy transformed after integration by parts and $\rho_2 = 1$. This Hamiltonian structure is the generalization of the canonical Zakharov form [13] to interfacial waves. These waves in linear approximation obey the dispersion relation

$$\omega_k = \left(\frac{|k|}{1+\rho}[g(1-\rho) + \sigma k^2]\right)^{1/2}, \quad (5)$$

which, for $\rho = 0$, becomes that for gravity-capillary waves. In the normal variables a_k, equations (4) are rewritten in the standard form [14]

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k^*}, \quad (6)$$

where the Hamiltonian can be expanded in infinite series with respect to powers of a_k and a_k^*. This expansion, $H = H_0 + H_{int}$, starts from the quadratic Hamiltonian $H_0 = \int \omega_k |a_k|^2 dk$. H_{int} is responsible for nonlinear wave interactions. The convergence of this series takes place if the characteristic slope angle is small: $k\eta \ll 1$. The simplest solution to equation (6) in the form of a stationary nonlinear wave depends on x and t in the combination $x - Vt$. For the normal variables it means an exponential dependence in time, $a_k(t) = c_k e^{-ikVt}$, since

$$\psi(x,t) \equiv \psi(x - Vt) = \frac{1}{\sqrt{2\pi}}\int a_k(t) e^{ikx} dk.$$

Here c_k is time-independent and defined from the equation

$$(\omega_k - kV)\, c_k = -\frac{\delta H_{int}}{\delta c_k^*} \equiv f_k. \quad (7)$$

The latter equation can be considered as an equation for stationary points of the Hamiltonian H for fixed momentum $P = \int k|c_k|^2 dk$:

$$\delta(H - VP) = 0. \quad (8)$$

Solutions to equation (7) in the form of stationary solitary waves are possible only when $\omega_k - kV$ is sign-definite, otherwise a solitary pulse will loose its energy due to Cherenkov radiation (see [11, 15]). For the given dispersion relation (5) the solitary wave velocity V must be less than the minimum phase velocity of internal linear waves,

$$V_{cr} = \min \frac{\omega_k}{k}.$$

As the velocity approaches V_{cr}, the amplitude c_k reaches a very sharp maximum at the point $k = k_0$, where the straight line $\omega = kV$ touches the dispersion curve $\omega = \omega_k$:

$$c_k \approx \left[\frac{1}{2}\omega'' \kappa^2 + k_0(V_{cr} - V)\right]^{-1} f_k. \quad (9)$$

Here $\kappa = k - k_0$ and $\omega'' = \partial^2\omega/\partial k^2 = \omega_0/2k_0^2 > 0$ is the positive-definite second derivative of ω_k at the point $k = k_0$, $\omega_0 \equiv \omega_{k_0}$. Due to nonlinearity this peak generates multiple harmonics near $k = nk_0$ with n an integer. If the amplitude of this peak is small (supercritical bifurcation or subcritical one with a small amplitude jump at $V = V_{cr}$), then the perturbation theory can be developed in the form of an expansion of ψ into harmonics:

$$\psi(x') = \sum_{n=-\infty}^{\infty} \psi_n(X) e^{ik_0 x'}, \quad x' = x - Vt. \quad (10)$$

Here the small parameter $\lambda = \sqrt{1 - V/V_{cr}}$ and the "slow" coordinate $X = \lambda x'$ are introduced, so that $\psi_n(X)$ is the amplitude of the envelope of n-th harmonic which is supposed to be small with respect to λ in comparison with the first one. Substituting this ansatz into Eq. (7) at leading order in λ yields the stationary NLSE:

$$-\lambda^2 \omega_0 \psi_1 + \frac{\omega_0}{4k_0^2}\frac{\partial^2 \psi_1}{\partial x^2} - \mu|\psi_1|^2\psi_1 = 0, \quad (11)$$

where the interaction Hamiltonian H_{int} is restricted to

$$\overline{H}^{(4)} = \frac{\widetilde{T}_{k_0k_0k_0k_0}}{2} \times \int c_k^* c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3}\, dk_1 dk_2 dk_3 dk_4 = = \frac{\mu}{2}\int |\psi|^4\, dx. \quad (12)$$

Here the subscript 1 for ψ_1 is dropped, and $\widetilde{T}_{k_0k_0k_0k_0}$ is the renormalization of the vertex T due to the interaction with the zeroth and second harmonics, corresponding to the cubic terms in the Hamiltonian. For the given case of interfacial waves,

$$\mu = \frac{k_0^3}{1+\rho}\left(A_{cr}^2 - A^2\right),$$

where $A = (1-\rho)/(1+\rho)$ is the Atwood number, and $A_{cr}^2 = 5/16$ or $\rho_{cr} = (21 - 8\sqrt{5})/11$ [4]. Below ρ_{cr}, the coupling coefficient μ is negative, the nonlinearity is of focusing type and solitons undergo a *supercritical* bifurcation: their amplitude vanishes proportionally to $\sqrt{V_{cr} - V}$ while their width increases as $(V_{cr} - V)^{-1/2}$. Thus, the NLS approximation improves as the bifurcation point is approached and becomes exact at $V = V_{cr}$. This type of bifurcation also takes place for gravity-capillary waves ($\rho = 0$). It was observed first in the numerical experiments of Longuet-Higgins [1] and explained later in [2–6].

3. For $\rho > \rho_{cr}$, the nonlinearity in (11) becomes defocusing ($\mu > 0$) and therefore, in order to get soliton solutions, one needs to keep next order terms beyond the classical NLS. If ρ is close to its critical value, solitons retain their envelope character. The next order terms come from the expansion of four-wave matrix element $\widetilde{T}_{k_1k_2k_3k_4}$ near the point $k_i = k_0$ and from six-wave interactions. The former contains local terms proportional to β and nonlocal terms proportional to γ (see [18]):

$$\widetilde{T}_{k_1k_2k_3k_4} = \frac{\mu}{2\pi} + \frac{\beta}{2\pi}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) - \\ - \frac{\gamma}{8\pi}(|\kappa_1 - \kappa_3| + |\kappa_2 - \kappa_3| + |\kappa_2 - \kappa_4| + |\kappa_1 - \kappa_4|),$$

where $\kappa_i = k_i - k_0$ and

$$\gamma = \frac{k_0^2}{(1+\rho)}, \quad \beta = \frac{3k_0^2}{16(1+\rho)}.$$

They yield two additional contributions in $\overline{H}^{(4)}$ (12):

$$\overline{H}^{(4)} = \frac{1}{2}\int \left[\mu|\psi|^4 + 2i\beta(\psi_x^*\psi - \psi_x\psi^*)|\psi|^2 - \gamma|\psi|^2\widehat{k}|\psi|^2\right]dx.$$

Here $\widehat{k} = -\partial_x \widehat{H}$ is the positive definite integral operator and $\widehat{H}$ is the Hilbert transform

$$\hat{H}f(x) = \frac{1}{\pi}\left(\text{P.V.}\int_{-\infty}^{\infty} \frac{f(x')dx'}{x' - x}\right).$$

The second term is analogous to the Lifshitz term in phase transition theory [10]. The nonlocal contribution is associated with the interaction of the soliton with the low frequency motion induced by the soliton packet. For $\rho = 0$ the nonlocal term coincides with that first obtained by Dysthe for gravity waves [12]. Due to the positiveness of γ the nonlocal interaction is of focusing type and can provide wave localization. Calculations show (for details see [18]) that the contribution from six-wave interactions is also of the focusing type:

$$\overline{H}^{(6)} = -C\int |\psi|^6\, dx,$$

where the coupling coefficient C is found after taking into account all renormalizations due to three-, four- and five-wave interactions :

$$C = \frac{Mk_0^6}{\omega_0}, \quad M = \frac{289(21 + 8\sqrt{5})}{49152} \approx 0.228654.$$

The soliton envelope is obtained from the solution of the following variational problem that arises as a result of averaging the problem (8) over the 'fast' spatial oscillations:

$$\delta(\overline{H} + \lambda^2\omega_0 N) = 0, \tag{13}$$

where the (averaged) Hamiltonian $\overline{H}$ and the number of waves N are given by the expressions:

$$\overline{H} = \frac{1}{2}\int \omega''|\psi_x|^2 dx + \overline{H}^{(4)} + \overline{H}^{(6)}, \quad N = \int |\psi|^2\, dx. \tag{14}$$

The variational problem (13) leads to

$$-\lambda^2\omega_0\psi + \frac{\omega_0}{4k_0^2}\psi_{xx} - \mu|\psi|^2\psi + 4i\beta|\psi|^2\psi_x + \\ + \gamma\psi\hat{k}|\psi|^2 + 3C|\psi|^4\psi = 0. \tag{15}$$

By introducing amplitude and phase, $\psi = re^{i\varphi}$, and separating real and imaginary parts in (15), it is easy to see that the phase is expressed through the amplitude,

$$\varphi_x = -\beta\frac{4k_0^2}{\omega_0}r^2,$$

and therefore it can be excluded from the equation for the amplitude:

$$-\lambda^2\omega_0 r + \frac{\omega_0}{4k_0^2}r_{xx} - \mu r^3 + \gamma r\hat{k}(r^2) + 3C_1 r^5 = 0, \tag{16}$$

where the six-wave coupling coefficient C is renormalized: $C_1 = C + \frac{4k_0^2}{\omega_0}\beta^2$. In dimensionless variables

$$x = x'\frac{\sqrt{3\omega_0 C_1}}{2k_0|\mu|}, \quad r = r'\sqrt{\frac{|\mu|}{3C_1}}, \\ \lambda' = \frac{\lambda}{|\mu|}\sqrt{3C_1\omega_0}, \quad \gamma' = \gamma\frac{2k_0}{\sqrt{3\omega_0 C_1}} = \frac{32}{\sqrt{397}}, \tag{17}$$

Eq. (16) can be rewritten in the form

$$-\lambda^2 r + r_{xx} - \mu r^3 + r^5 + \gamma r \hat{k}(r^2) = 0, \qquad (18)$$

where all primes are omitted and $\mu = \mathrm{sign}(\rho - \rho_{cr})$.

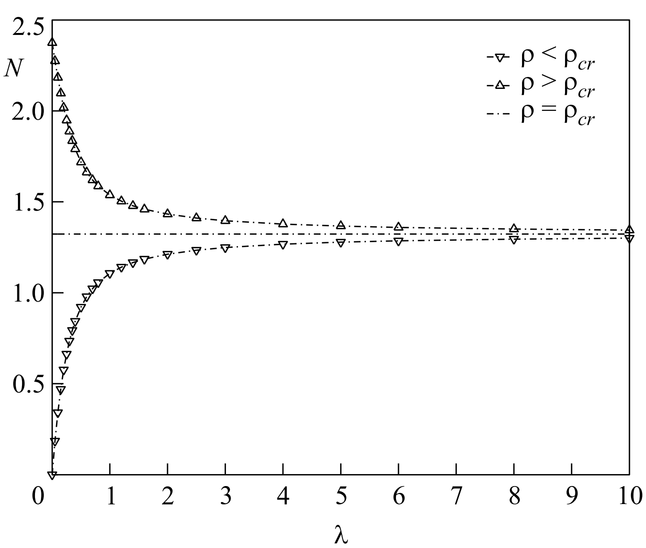
Below the critical density ratio ($\mu = -1$) all nonlinear terms in (18) are of the focusing type. As a result solitons exist in the whole range of λ and we arrive at a supercritical bifurcation: the soliton amplitude vanishes like $\lambda^{1/2}$ as $\lambda \to 0$.

Above the critical density ratio solitons undergo a *subcritical* bifurcation. In this case, at $\lambda = 0$, a soliton solution with finite amplitude can be found explicitly. It has the form $r = A\left(x^2 + a^2\right)^{-1/2}$ where the soliton amplitude A and its width a are given by the expressions:

$$A^2 = \gamma^2 + 2 - \gamma\sqrt{\gamma^2 + 3}, \quad a = \frac{1}{3}\left(-\gamma + 2\sqrt{\gamma^2 + 3}\right). \qquad (19)$$

In the dimensional variables (17), the soliton amplitude is proportional to $\sqrt{\rho - \rho_{cr}}$. Hence immediately we have the applicability criterion of our theory: $\rho - \rho_{cr} \ll \rho_{cr}$.

The algebraic soliton (19) represents the boundary for the whole soliton family decaying at infinity like $e^{-\lambda|x|}$. Solitons of this family can be found numerically by means of the Petviashvili scheme [19]. Figure shows the dependence of the number of waves versus λ for both



Dependence of N on λ for $\mu = 0, \pm 1$

soliton branches ($\mu \pm 1$). For large values of λ the soliton solutions are asymptotically obtained from the equation

$$-\lambda^2 r + r_{xx} + \gamma r \widehat{k} r^2 + r^5 = 0, \qquad (20)$$

and these families converge. The asymptotic equation (20) can be related to the so-called critical NLS equation (at $\gamma = 0$ this fact is well known, see e.g. [17]). In this case the number of waves $N = N_{cr}$ does not depend on λ. For solitons above ρ_{cr} the number of waves approaches N_{cr} from above and for the soliton branch with $\mu = -1$ we have convergence from below.

4. The dependence $N(\lambda)$ allows to make some predictions about the soliton stability based on the Vakhitov-Kolokolov criterion [20]. This criterion states that if the derivative $\partial N_s/\partial\lambda^2$ along the soliton family is negative then solitons are unstable and they are stable in the opposite situation. The physical meaning of the quantity $-\lambda^2$ is related to the energy of a soliton as a bound state. If by adding a particle (that means increasing N) this level shifts towards the continuous spectrum, then such state is unstable. Therefore we should expect stability of solitons below ρ_{cr}. It is in agreement with the stability of the classical NLS solitons [17] considered as the limiting case of the lower soliton branch as $\lambda \to 0$. Respectively, this criterion gives instability of solitons with $\mu > 0$. However, the Vakhitov-Kolokolov criterion, derived for the classical NLSE, cannot be applied to our system strictly speaking.

In order to estimate soliton stability we will use the Lyapunov theorem and consider the dependence of the Hamiltonian $\overline{H}$ (14) evaluating on the class of functions with N fixed. The scaling transformation

$$r = \frac{1}{a^{1/2}} r_s\left(\frac{x}{a}\right),$$

applied to a soliton solution r_s of Eq. (16), makes $\overline{H}$ a function of the scaling parameter a:

$$\overline{H}(a) = \left(\frac{1}{a} - \frac{1}{2a^2}\right)\frac{\mu}{2}\int r_s^4 dx;$$

$$\overline{H}_s \equiv \overline{H}(a = 1) = \frac{1}{4}\int \mu r^4 dx.$$

Hence, for $\mu < 0$, the function $\overline{H}(a)$ is bounded from below and its minimum corresponds to the soliton solution with $\overline{H}_s < 0$. For $\mu > 0$, this function is unbounded from below as $a \to 0$ and has a maximum $\overline{H}_s > 0$ corresponding to the soliton solution. Another transformation of the gauge type, $\psi = \psi_s e^{i\chi}$, demonstrates that this soliton represents a saddle point of the Hamiltonian and therefore the upper soliton branch is unstable (at least, with respect to finite perturbations). For the lower soliton branch both these transformations give a minimum. Now we show that the lower branch of solitons indeed achieves a minimum of the Hamiltonian $\overline{H}$ for fixed N. The Hamiltonian $\overline{H}$ (14) written in terms of amplitude and phase reads

$$\overline{H} = \int \left[r_x{}^2 + \frac{\mu}{2}r^4 - \frac{\gamma}{2}r^2\widehat{k}r^2 - \frac{1}{3}r^6 + r^2\left(\varphi_x + \beta r^2\right)^2\right] dx.$$

The last term here is positive definite and vanishes exactly on the soliton solutions. The fourth integral can be estimated as follows:

$$\int r^6 dx \le \left(\frac{N}{N_1}\right)^2 \int {r_x}^2\, dx,$$

with $N_1 = \pi/2$ [11]. For the integral $I_k = \int r^2 \widehat{k} r^2 dx$, we can write the following set of inequalities:

$$\int r^2 \widehat{k} r^2 dx \le \max_x \left(r^2\right) \int r \widehat{k} r\, dx \le$$
$$\le \int_{-\infty}^{x_{\max}} \left(r^2\right)_x dx \left(\int r^2 dx \int r \widehat{k}^2 r\, dx\right)^{1/2} \le$$
$$\le C_2 N \int {r_x}^2\, dx.$$

This inequality can be made sharper by finding the best constant C_2 which coincides with the minimum value for the functional $F\{r\} = \int r^2 \widehat{k} r^2 dx \left(\int r^2 dx \cdot \int {r_x}^2 dx\right)^{-1}$. The corresponding minimizer is obtained for the ground soliton solution of the equation

$$2 r_0 \widehat{k} r_0^2 - r_0 + r_{0xx} = 0.$$

It gives $C_{2,\text{best}} = 1/N_2$. Here $N_2 = \int r_0^2 dx \approx 1.39035$. According to [15] $\int r^4 dx \le \frac{1}{\sqrt{3}} N^{3/2} \left(\int {r_x}^2 dx\right)^{1/2}$. Applying all these inequalities gives the Hamiltonian boundedness from below:

$$\overline{H} \ge -\frac{N^3}{4\sqrt{3}} \left[1 - \left(\frac{\gamma}{2}\frac{N}{N_2} + \frac{N^2}{3N_1^2}\right)\right],$$

if

$$N \le N_3 = \sqrt{\frac{9}{16}\gamma^2 \frac{N_1^4}{N_2^2} + 3N_1^2} - \frac{3}{4}\gamma \frac{N_1^2}{N_2}.$$

Thus, in accordance with the Lyapunov theorem, solitons from the lower branch with $N \le N_3$ are stable not only with respect to small perturbations but also against finite ones. The numerical value of $N_3 = 1.3224$ is almost the same as the critical number $N_{cr} = 1.3225$, defined by solitons of (20).

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1. M. S. Longuet-Higgins, J. Fluid Mech. **200**, 451 (1989).
2. G. Iooss and K. Kirchgässner, C. R. Acad. Sci. Paris **311**(1), 265 (1991).
3. J.-M. Vanden-Broeck and F. Dias, J. Fluid Mech. **240**, 549 (1992).
4. F. Dias and G. Iooss, Physica D **65**, 399(1993).
5. M. S. Longuet-Higgins, J. Fluid Mech. **252**, 703 (1993).
6. T. R. Akylas, Phys. Fluids **5**, 789 (1993).
7. F. Dias and E. A. Kuznetsov, Phys. Letters A **263**, 98 (1999).
8. E. Falcon, C. Laroche, and S. Fauve, Phys. Rev. Lett. **89**, 204501 (2002).
9. F. Dias and G. Iooss, Eur. J. Mech. B/Fluids **15**, 367 (1996).
10. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part 1, Pergamon Press, New York [Russian original: Nauka, Moscow, 1995, p. 521].
11. E. A. Kuznetsov, ZhETF **116**, 299 (1999) [JETP **89**, 163 (1999)].
12. K. B. Dysthe, Proc. R. Soc. Lond. A **369**, 105 (1979).
13. V. E. Zakharov, Zh. Prikl. Mekh. Tekh. Fiz. **9**, 86 (1968) [J. Appl. Mech. Tech. Phys. **9**, 190 (1968)].
14. V. E. Zakharov and E. A. Kuznetsov, Usp. Fiz. Nauk **167**, 1137 (1997) [Physics-Uspekhi **40**, 1087 (1997)].
15. V. E. Zakharov and E. A. Kuznetsov, ZhETF **113**, 1892 (1998) [JETP **86**, 1035 (1998)].
16. V. M. Kontorovich, Izv.Vuzov, Radiofizika, **19**, 872 (1976); E. A. Kuznetsov and M. D. Spector, ZhETF **71**, 262 (1976) [Sov. Phys. JETP **44**, 136 (1976)].
17. E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142**, 103 (1986).
18. D. S. Agafontsev, F. Dias, and E. A. Kuznetsov, physics/0512006.
19. V. I. Petviashvili, Fizika Plasmy **2**, 469 (1976) [Sov. J. Plasma Phys. **2**, 247 (1976)].
20. I. G. Vakhitov and A. A. Kolokolov, Izv. VUZ Radiofizika **16**, 1020 (1973) [Radiophys. Quant. Electron. **16**, 783 (1973)].