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The Cauchy Problem on the Plane for the Dispersionless Kadomtsev – Petviashvili Equation

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We construct the formal solution of the Cauchy problem for the dispersionless Kadomtsev–Petviashvili equation as application of the Inverse Scattering Transform for the vector field corresponding to a Newtonian particle in a time-dependent potential. This is in full analogy with the Cauchy problem for the Kadomtsev–Petviashvili equation, associated with the Inverse Scattering Transform of the time dependent Schrödinger operator for a quantum particle in a time-dependent potential.

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1. Dispersionless (or quasi-classical) limits of integrable partial differential equations (PDEs) arise in various problems of Mathematical Physics and are intensively studied in the recent literature (see, f.i., [1–5]). In particular, a quasi-classical dressing has been developed for the prototypical example of the dispersionless Kadomtsev - Petviashvili (dKP) (or Khokhlov-Zabolotskaya) equation:

$$u_{tx} + u_{yy} + (uu_x)_x = 0,\ u = u(x,y,t) \in \mathbb{R},\ x,y,t \in \mathbb{R}. \qquad (1)$$

In this paper we construct the formal solution of the Cauchy problem on the plane for the following system of PDEs in 2+1 dimensions:

$$\begin{gathered} u_{xt} + u_{yy} = -(uu_x)_x - v_x u_{xy} + v_y u_{xx}, \\ u, v \in \mathbb{R}, \quad x,y,t \in \mathbb{R}, \qquad (2) \\ v_{xt} + v_{yy} = -uv_{xx} - v_x v_{xy} + v_y v_{xx} \end{gathered}$$

and for its $v = 0$ reduction, the dKP Eq. (1), as application of the recently developed Inverse Scattering Transform (IST) for vector fields [6]. Indeed the system (2) arises as the compatibility condition of the Lax pair

$$\widehat{L}_1 \psi = 0, \quad \widehat{L}_2 \psi = 0, \qquad (3)$$

implying $[\widehat{L}_1, \widehat{L}_2] = 0$, where $\widehat{L}_1,\ \widehat{L}_2$ are the following vector fields:

$$\begin{gathered} \widehat{L}_1 \equiv \partial_y + (p + v_x)\partial_x - u_x \partial_p, \\ \widehat{L}_2 \equiv \partial_t + (p^2 + pv_x + u - v_y)\partial_x + (-pu_x + u_y)\partial_p. \end{gathered} \qquad (4)$$

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Setting $v = 0$ in (4), one obtains the Lax pair of the dKP equation, which was derived in [3] taking the quasi-classical limit of the well-known Lax pair of the KP equation [7, 8].

We remark that, in the dKP reduction $v = 0$, the two vector fields are Hamiltonian and the Lax pair (4) takes the form

$$\begin{gathered} \psi_y + p\psi_x - u_x \psi_p = \psi_y + \{H_1, \psi\}_{(p,x)} = 0, \\ \psi_t + (p^2 + u)\psi_x + (-pu_x + u_y)\psi_p = \\ = \psi_t + \{H_2, \psi\}_{(p,x)} = 0, \end{gathered} \qquad (5)$$

in terms of the two Hamiltonians [3]

$$H_1 = \frac{p^2}{2} + u, \qquad H_2 = \frac{p^3}{3} + pu - \partial_x^{-1} u_y, \qquad (6)$$

where $\{\cdot,\cdot\}_{(p,x)}$ is the standard Poisson bracket with respect to the canonical variables (p,x):

$$\{f, g\}_{(p,x)} \equiv f_p g_x - f_x g_p, \qquad (7)$$

leading to the Hamiltonian form of dKP: $H_{1t} - H_{2y} + \{H_1, H_2\}_{(p,x)} = 0$.

Since the Lax pair (3) of the dKP-like system (2) is made of vector fields, Hamiltonian in the dKP reduction (1), the eigenfunctions satisfy the following basic properties.

1) *The space of eigenfunctions is a ring.* If $f_1,\ f_2$ are two solutions of the Lax pair (3), then an arbitrary differentiable function $F(f_1, f_2)$ of them is a solution of (3).

2) *In the dKP reduction $v = 0$, the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket (7).* If $f_1,\ f_2$ are two solutions of

the Lax pair (5), then their Poisson bracket $\{f_1, f_2\}_{(p,x)}$ is also a solution of (5).

2. Now we consider the Cauchy problem for the dKP system (2) and for the dKP equation (1) within the class of rapidly decreasing real potentials u, v:

$$u, v \to 0,\ (x^2+y^2) \to \infty,\ u, v \in \mathbb{R},\ (x,y) \in \mathbb{R}^2,\ t > 0, \tag{8}$$

interpreting t as time and the other two variables x, y as space variables. To solve such a Cauchy problem by the IST method [9], we construct the IST for the operator $\widehat{L}_1$, within the class of rapidly decreasing real potentials, interpreting the operator $\widehat{L}_2$ as the time operator.

The localization (8) of the potentials u, v implies that, if f is a solution of $\widehat{L}_1 f = 0$, then

$$\begin{gathered} f(x,y,p) \to f_\pm(\xi,p),\ \ y \to \pm\infty, \\ \xi := x - py; \end{gathered} \tag{9}$$

i.e., asymptotically, f is an arbitrary function of $\xi = x - py$ and p.

A central role in the theory is played by the two real Jost eigenfunctions $\varphi_{1,2}(x,y,p)$, the solutions of $\widehat{L}_1\varphi_{1,2} = 0$ uniquely defined by the asymptotics

$$\varphi_1(x,y,p) \to \xi, \quad \varphi_2(x,y,p) \to p, \quad y \to -\infty. \tag{10}$$

In this paper we often use the compact vector notation: $\mathbf{f} = (f_1, f_2)^T$. Then:

$$\boldsymbol{\varphi}(x,y,p) \equiv \begin{pmatrix} \varphi_1(x,y,p) \\ \varphi_2(x,y,p) \end{pmatrix} \to \begin{pmatrix} \xi \\ p \end{pmatrix} \equiv \boldsymbol{\xi},\ \ y \to -\infty. \tag{11}$$

The Jost eigenfunction $\boldsymbol{\varphi}$ is the solution of the linear integral equations $\boldsymbol{\varphi} = \boldsymbol{\xi} + \widehat{G}(-v_x\boldsymbol{\varphi}_x + u_x\boldsymbol{\varphi}_p)$, for the Green's function $G(x,y,p) = \theta(y)\delta(x-py)$.

The $y = +\infty$ limit of $\boldsymbol{\varphi}$ defines the natural scattering vector $\boldsymbol{\sigma}$ for $\widehat{L}_1$:

$$\lim_{y\to+\infty}\boldsymbol{\varphi}(x,y,p) \equiv \boldsymbol{\mathcal{S}}(\boldsymbol{\xi}) = \boldsymbol{\xi} + \boldsymbol{\sigma}(\boldsymbol{\xi}). \tag{12}$$

The direct problem is the transformation from the real potentials u, v, functions of the two real variables (x,y), to the two real scattering data σ_1, σ_2, the components of the scattering vector $\boldsymbol{\sigma}$, functions of the two real variables (ξ, p). Therefore the mapping is consistent. The impact of the dKP reduction $v = 0$ on these and other data will be shown below.

A crucial role in the IST theory for the vector field $\widehat{L}_1$ is also played by the analytic eigenfunctions $\boldsymbol{\psi}_\pm(x,y,p)$, the solutions of $\widehat{L}_1\boldsymbol{\psi}_\pm = \mathbf{0}$ satisfying the integral equations

$$\begin{aligned} \boldsymbol{\psi}_\pm(x,y,p) = \int_{\mathbb{R}^2} dx'dy' G_\pm(x-x', y-y', p) \times \\ \times [-v_{x'}(x',y')\boldsymbol{\psi}_{\pm_{x'}}(x',y',p) + \\ + u_{x'}(x',y')\boldsymbol{\psi}_{\pm_p}(x',y',p)] + \boldsymbol{\xi}, \end{aligned} \tag{13}$$

where $G_\pm$ are the analytic Green's functions

$$G_\pm(x,y,p) = \pm\frac{1}{2\pi i[x - (p \pm i\epsilon)y]}. \tag{14}$$

The analyticity properties of $G_\pm(x,y,p)$ in the complex p-plane imply that $\boldsymbol{\psi}_+(x,y,p)$ and $\boldsymbol{\psi}_-(x,y,p)$ are analytic, respectively, in the upper and lower halves of the p-plane, with the following asymptotics, for large p:

$$\begin{gathered} \boldsymbol{\psi}_\pm(x,y,p) = \boldsymbol{\xi} + \tfrac{1}{p}\mathbf{U}(x,y) + \mathbf{O}\left(\tfrac{1}{p^2}\right),\ \ |p| \gg 1, \\ \mathbf{U}(x,y) \equiv \begin{pmatrix} -yu(x,y) - v(x,y) \\ u(x,y) \end{pmatrix}. \end{gathered} \tag{15}$$

It is important to remark that the analytic Green's functions (14) exhibit the following asymptotics for $y \to \pm\infty$:

$$\begin{aligned} G_\pm(x-x', y-y', p) \to \pm\frac{1}{2\pi i[\xi - \xi' \mp i\epsilon]},\ \ y \to +\infty, \\ G_\pm(x-x', y-y', p) \to \pm\frac{1}{2\pi i[\xi - \xi' \pm i\epsilon]},\ \ y \to -\infty, \end{aligned} \tag{16}$$

entailing that *the $y = +\infty$ asymptotics of $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are analytic respectively in the lower and upper halves of the complex plane ξ, while the $y = -\infty$ asymptotics of $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are analytic respectively in the upper and lower halves of the complex plane ξ* (similar features have been observed first in [10] and later in [6]).

The Jost eigenfunctions $\varphi_{1,2}$ form a basis; thus any solution f of $\widehat{L}_1 f = 0$ is a function of $\boldsymbol{\varphi}$. The analytic eigenfunctions $\boldsymbol{\psi}_\pm$ possess the representations:

$$\boldsymbol{\psi}_\pm = \boldsymbol{\mathcal{K}}_\pm(\boldsymbol{\varphi}) = \boldsymbol{\varphi} + \boldsymbol{\chi}_\pm(\boldsymbol{\varphi}), \tag{17}$$

defining the spectral data $\boldsymbol{\chi}_\pm$.

Since the $y \to -\infty$ limit of (17) read:

$$\lim_{y\to-\infty}\boldsymbol{\psi}_\pm - \boldsymbol{\xi} = \boldsymbol{\chi}_\pm(\boldsymbol{\xi}), \tag{18}$$

the above analyticity properties of the LHS of (18) in the complex ξ-plane imply that $\boldsymbol{\chi}_+(\boldsymbol{\xi})$ and $\boldsymbol{\chi}_-(\boldsymbol{\xi})$ are analytic respectively in the upper and lower halves of the complex plane ξ, decaying at $\xi \sim \infty$ like $O(\xi^{-1})$. Therefore their Fourier transforms $\tilde{\boldsymbol{\chi}}_+(\boldsymbol{\omega})$ and $\tilde{\boldsymbol{\chi}}_-(\boldsymbol{\omega})$ have support respectively on the positive and negative ω_1 semi-axes.

The spectral vectors $\boldsymbol{\chi}_\pm$ can be constructed from the scattering vector $\boldsymbol{\sigma}$ through the following linear integral equations

$$\begin{gathered}\tilde{\boldsymbol{\chi}}_+(\boldsymbol{\omega})+\theta(\omega_1)\left(\tilde{\boldsymbol{\sigma}}(\boldsymbol{\omega})+\textstyle\int_{\mathbb{R}^2}d\boldsymbol{\eta}\;\tilde{\boldsymbol{\chi}}_+(\boldsymbol{\eta})Q(\boldsymbol{\eta},\boldsymbol{\omega})\right)=0,\\ \tilde{\boldsymbol{\chi}}_-(\boldsymbol{\omega})+\theta(-\omega_1)\left(\tilde{\boldsymbol{\sigma}}(\boldsymbol{\omega})+\textstyle\int_{\mathbb{R}^2}d\boldsymbol{\eta}\;\tilde{\boldsymbol{\chi}}_-(\boldsymbol{\eta})Q(\boldsymbol{\eta},\boldsymbol{\omega})\right)=0,\end{gathered}\tag{19}$$

involving the Fourier transforms $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\chi}}_\pm$ of $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}_\pm$:

$$\tilde{\boldsymbol{\sigma}}(\boldsymbol{\omega})=\int_{\mathbb{R}^2}d\boldsymbol{\xi}\boldsymbol{\sigma}(\boldsymbol{\xi})e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\xi}},\quad \tilde{\boldsymbol{\chi}}_\pm(\boldsymbol{\omega})=\int_{\mathbb{R}^2}d\boldsymbol{\xi}\boldsymbol{\chi}_\pm(\boldsymbol{\xi})e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\xi}}\tag{20}$$

and the kernel:

$$Q(\boldsymbol{\eta},\boldsymbol{\omega})=\int_{\mathbb{R}^2}\frac{d\boldsymbol{\xi}}{(2\pi)^2}e^{i(\boldsymbol{\eta}-\boldsymbol{\omega})\cdot\boldsymbol{\xi}}[e^{i\boldsymbol{\eta}\cdot\boldsymbol{\sigma}(\boldsymbol{\xi})}-1].\tag{21}$$

To prove this result, one first evaluates (17) at $y=+\infty$, obtaining

$$\left(\lim_{y\to\infty}\boldsymbol{\psi}_\pm-\boldsymbol{\xi}\right)-\boldsymbol{\sigma}(\boldsymbol{\xi})=\boldsymbol{\chi}_\pm(\boldsymbol{\xi}+\boldsymbol{\sigma}(\boldsymbol{\xi})).\tag{22}$$

Applying the integral operator $\int_{\mathbb{R}^2}d\boldsymbol{\xi}e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\xi}}\cdot$ for $\omega_1>0$ and $\omega_1<0$ respectively to equations $(22)_+$ and $(22)_-$, using the above analyticity properties and the Fourier representations of $\boldsymbol{\chi}_\pm$ and $\boldsymbol{\sigma}$, one obtains equations (19).

The reality of the potentials: $u,v\in\mathbb{R}$ implies that, for $p\in\mathbb{R}$, $\overline{\boldsymbol{\varphi}}=\boldsymbol{\varphi}$, $\overline{\boldsymbol{\psi}}_+=\boldsymbol{\psi}_-$; consequently: $\overline{\boldsymbol{\sigma}}=\boldsymbol{\sigma}$, $\overline{\boldsymbol{\chi}}_+=\boldsymbol{\chi}_-$.

3. An inverse problem can be constructed from Eqs. (17). Subtracting $\boldsymbol{\xi}$ from Eqs. $(17)_-$ and $(17)_+$, applying respectively the analyticity projectors $\widehat{P}_+$ and $\widehat{P}_-$:

$$\widehat{P}_\pm\equiv\pm\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{dp'}{p'-(p\pm i\epsilon)}.\tag{23}$$

and adding up the resulting equations, one obtains the following nonlinear integral equation for the Jost eigenfunction $\boldsymbol{\varphi}$:

$$\begin{gathered}\boldsymbol{\varphi}(x,y,p)+\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{dp'}{p'-(p+i\epsilon)}\boldsymbol{\chi}_-(\boldsymbol{\varphi}(x,y,p'))-\\ -\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{dp'}{p'-(p-i\epsilon)}\boldsymbol{\chi}_+(\boldsymbol{\varphi}(x,y,p'))=\boldsymbol{\xi}.\end{gathered}\tag{24}$$

Once $\boldsymbol{\varphi}$ is reconstructed from (24), the analytic eigenfunctions follow from (17), and u,v from Eq. (15). This inversion procedure was first introduced in [11] and also used in [6].

4. As u,v evolve in time according to (2), the t-dependence of the spectral data $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{K}}_\pm$, defined in (12) and (17), is described by the equations:

$$\begin{gathered}\Sigma_1(\xi,p,t)=t\left(\Sigma_2(\xi-p^2t,p,0)\right)^2+\Sigma_1(\xi-p^2t,p,0),\\ \Sigma_2(\xi,p,t)=\Sigma_2(\xi-p^2t,p,0),\end{gathered}\tag{25}$$

where Σ_1 and Σ_2 are the two components of the vector $\boldsymbol{\Sigma}$, identifiable with each of the spectral vectors $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{K}}_\pm$. To prove it, we first observe that

$$\begin{gathered}\phi_1(x,y,t,p)\equiv\varphi_1(x,y,t,p)-t\varphi_2^2(x,y,t,p),\\ \phi_2(x,y,t,p)\equiv\varphi_2(x,y,t,p)\end{gathered}\tag{26}$$

are a basis of common Jost eigenfunctions of $\widehat{L}_1$ and $\widehat{L}_2$. The $y=+\infty$ limit of equation $\widehat{L}_2\phi_2=0$ yields $\mathcal{S}_{2t}+p^2\mathcal{S}_{2\xi}=0$, while the $y=+\infty$ limit of equation $\widehat{L}_2\phi_1=0$ yields $(\partial_t+p^2\partial_\xi)(\mathcal{S}_1-t\mathcal{S}_2^2)=0$, whose solutions are (25) for $\boldsymbol{\mathcal{S}}$. Analogously,

$$\begin{gathered}\pi_{\pm1}(x,y,t,p)\equiv\psi_{\pm1}(x,y,t,p)-t\psi_{\pm2}^2(x,y,t,p),\\ \pi_{\pm2}(x,y,t,p)\equiv\psi_{\pm2}(x,y,t,p)\end{gathered}\tag{27}$$

are a basis of common analytic eigenfunctions of $\widehat{L}_1$ and $\widehat{L}_2$; therefore

$$\pi_{\pm1}=\check{\mathcal{K}}_{\pm1}(\phi_1,\phi_2),\quad \pi_{\pm2}=\check{\mathcal{K}}_{\pm2}(\phi_1,\phi_2),\tag{28}$$

for some functions $\check{\mathcal{K}}_{\pm1,2}$ of just two arguments (not depending explicitly on t). Comparing at $t=0$ these equations with Eqs. (17), one expresses $\check{\mathcal{K}}_{\pm1,2}$ in terms of $\mathcal{K}_{\pm1,2}$, obtaining Eqs. (25) for $\mathcal{K}_{\pm1,2}$.

We observe the unusual resonant character of the explicit t-dependence (25) of the spectral data, if compared to the more elementary one, obtained in [6], for the heavenly Eq. [12].

5. In the Hamiltonian dKP reduction $v=0$, the transformations $\boldsymbol{\xi}\to\boldsymbol{\mathcal{S}}(\boldsymbol{\xi})$, $\boldsymbol{\xi}\to\boldsymbol{\mathcal{K}}_\pm(\boldsymbol{\xi})$ are constrained to be canonical:

$$\{\mathcal{S}_1,\mathcal{S}_2\}_{(\xi,p)}=\{\mathcal{K}_{\pm1},\mathcal{K}_{\pm2}\}_{(\xi,p)}=1.\tag{29}$$

To prove it, we observe that the Poisson bracket of the eigenfunctions φ_1 and φ_2 is also an eigenfunction: $\varphi_3\equiv\{\varphi_1,\varphi_2\}_{(x,p)}$, $\widehat{L}_1\varphi_3=0$. Using the asymptotics (10), one infers that $\varphi_3\to1$, at $y\to-\infty$; therefore, by uniqueness, $\varphi_3=1$. Evaluating now the Poisson bracket φ_3 at $y=+\infty$ and using (12), one obtains the constraint (29) for $\boldsymbol{\mathcal{S}}$. We also observe that the eigenfunctions $\{\psi_{+1},\psi_{+2}\}_{(x,p)}$ and $\{\psi_{-1},\psi_{-2}\}_{(x,p)}$ are analytic in the upper and lower p plane and go to 1 at $|p|\to\infty$.

Since 1 is also an eigenfunction, by uniqueness they are identically 1: $\{\psi_{\pm 1}, \psi_{\pm 2}\}_{(x,p)} = 1$. Therefore, from the equations:

$$\{\psi_{\pm 1}, \psi_{\pm 2}\}_{(x,p)} = \{\mathcal{K}_{\pm 1}, \mathcal{K}_{\pm 2}\}_{(\varphi_1,\varphi_2)}\{\varphi_1, \varphi_2\}_{(x,p)} = 1, \tag{30}$$

consequence of (17), one infers the constraints (29).

6. It is well-known (see, f.i., [13]) that linear first order PDEs like (3),(4) are intimately related to systems of ordinary differential equations describing their characteristic curves. The Hamiltonian dynamical systems associated with the vector fields $\widehat{L}_{1,2}$ of dKP are:

$$\widehat{L}_1 : \quad \begin{cases} \dfrac{dx}{dy} = p = \{H_1, x\}_{(p,x)} \\ \dfrac{dp}{dy} = -u_x = \{H_1, p\}_{(p,x)} \end{cases}, \tag{31}$$

$$\widehat{L}_2 : \quad \begin{cases} \dfrac{dx}{dt} = p^2 + u = \{H_2, x\}_{(p,x)} \\ \dfrac{dp}{dt} = -pu_x + u_y = \{H_2, p\}_{(p,x)} \end{cases}. \tag{32}$$

Therefore *the dKP equation characterizes the class of time – dependent potentials for wich the Newtonian flow (31) commutes with a flow with cubic, in the momentum p, Hamiltonian.*

There is also a deep connection between the above IST and the y-scattering theory for the commuting flows (31) and (32). Let $\boldsymbol{\phi}(x,y,t,p)$ be the basis of common eigenfunctions of $\widehat{L}_1$ and $\widehat{L}_2$ defined in (26); then, solving the system $\boldsymbol{\omega} = \boldsymbol{\phi}(x,y,t,p)$ with respect to x and p (assuming local invertibility), one obtains the following common solution of (31) and (32):

$$\boldsymbol{\omega} = \boldsymbol{\phi}(x,y,t,p) \quad \Leftrightarrow \quad \begin{pmatrix} x \\ p \end{pmatrix} = \mathbf{r}(y,t;\boldsymbol{\omega}) \sim$$
$$\sim \begin{pmatrix} \omega_2 y + \omega_2^2 t + \omega_1 \\ \omega_2 \end{pmatrix}, \quad y \sim -\infty. \tag{33}$$

The $y = +\infty$ limit of the solution $\mathbf{r}(y,t;\boldsymbol{\omega})$:

$$\begin{pmatrix} x \\ p \end{pmatrix} \sim \begin{pmatrix} \Omega_2(\boldsymbol{\omega}) y + \Omega_2^2(\boldsymbol{\omega}) t + \Omega_1(\boldsymbol{\omega}) \\ \Omega_2(\boldsymbol{\omega}) \end{pmatrix}, \quad y \sim +\infty \tag{34}$$

defines the scattering vector $\boldsymbol{\Delta}(\boldsymbol{\omega}) = \boldsymbol{\Omega}(\boldsymbol{\omega}) - \boldsymbol{\omega}$ of (31) and (32), which is connected to the IST data $\boldsymbol{\mathcal{S}}$ by inverting the system $\boldsymbol{\omega} = \boldsymbol{\mathcal{S}}(x - py - p^2 t, p)$ with respect to x and p:

$$\boldsymbol{\omega} = \boldsymbol{\mathcal{S}}(x - py - p^2 t, p) \quad \Leftrightarrow \quad \begin{pmatrix} x \\ p \end{pmatrix} =$$
$$= \begin{pmatrix} \Omega_2(\boldsymbol{\omega}) y + \Omega_2^2(\boldsymbol{\omega}) t + \Omega_1(\boldsymbol{\omega}) \\ \Omega_2(\boldsymbol{\omega}) \end{pmatrix}. \tag{35}$$

The transformation $\boldsymbol{\omega} \to \boldsymbol{\Omega}(\boldsymbol{\omega})$ is clearly canonical: $\{\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2\}_{(\omega_1,\omega_2)} = 1$.

Since the dynamical system (31) describes the motion of a Newtonian particle in the plane subjected to a generic time-dependent potential $u(x,y)$, as a byproduct of the IST of this paper one can reconstruct, from the scattering vector $\boldsymbol{\Delta}(\boldsymbol{\omega})$ of the dynamical system (31), the time dependent potential u.

Remark 1. The are two other ways to do the inverse problem. The first one is the linear version of the nonlinear problem (24), obtained *exponentiating the Jost and analytic eigenfunctions* used so far. Consider the following scalar functions:

$$\Phi(x,y,p;\boldsymbol{\alpha}) \equiv e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\varphi}(x,y,p)},$$
$$\Psi_\pm(x,y,p;\boldsymbol{\alpha}) \equiv e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\psi}_\pm(x,y,p)}, \quad \boldsymbol{\alpha} \in \mathbb{R}^2. \tag{36}$$

Due to the ring property of the space of eigenfunctions, also $\Phi(x,y,p;\boldsymbol{\alpha})$ and $\Psi_\pm(x,y,p;\boldsymbol{\alpha})$ are eigenfunctions; $\Phi(x,y,p;\boldsymbol{\alpha})$ is characterized by the asymptotics $\Phi \to \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\xi})$, $y \to -\infty$, while $\Psi_\pm(x,y,p;\boldsymbol{\alpha})$ are analytic respectively in the upper and lower halves of the p plane, with asymptotics: $\Psi_\pm = \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\xi})[1 + p^{-1}\boldsymbol{\alpha} \times \times \mathbf{U}(x,y) + O(p^{-2})]$.

Exponentiating the representation (17), one obtains the expansions of the analytic eigenfunctions $\Psi_\pm$ in terms of the Jost eigenfunction Φ:

$$\Psi_\pm(x,y,p;\boldsymbol{\alpha}) = \Phi(x,y,p;\boldsymbol{\alpha}) + $$
$$+ \int_{\mathbb{R}^2} d\boldsymbol{\beta} K_\pm(\boldsymbol{\alpha},\boldsymbol{\beta})\Phi(x,y,p;\boldsymbol{\beta}),$$
$$K_\pm(\boldsymbol{\alpha},\boldsymbol{\beta}) \equiv \int_{\mathbb{R}^2} \frac{d\boldsymbol{\xi}}{(2\pi)^2} e^{i(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\boldsymbol{\xi}}[e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\chi}_\pm(\boldsymbol{\xi})} - 1]. \tag{37}$$

Multiplying the Eqs. $(37)_+$ and $(37)_-$ by $\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\xi})$, subtracting 1, applying respectively $\widehat{P}_-$ and $\widehat{P}_+$, and adding the resulting equations, one obtains the following *linear integral equation* for Φ:

$$\Phi(p;\boldsymbol{\alpha}) + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{dp'}{p' - (p + i\epsilon)} \times$$
$$\times \int_{\mathbb{R}^2} d\boldsymbol{\beta} K_-(\boldsymbol{\alpha},\boldsymbol{\beta})\Phi(p';\boldsymbol{\beta}) e^{i\boldsymbol{\alpha}\cdot(\boldsymbol{\xi}(p) - \boldsymbol{\xi}(p'))} -$$
$$- \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{dp'}{p' - (p - i\epsilon)} \int_{\mathbb{R}^2} d\boldsymbol{\beta} K_+(\boldsymbol{\alpha},\boldsymbol{\beta})\Phi(p';\boldsymbol{\beta}) \times$$
$$\times e^{i\boldsymbol{\alpha}\cdot(\boldsymbol{\xi}(p) - \boldsymbol{\xi}(p'))} = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\xi}(p)}, \tag{38}$$

in which we have omitted, for simplicity, the parametric dependence on (x, y). Once Φ is reconstructed from (38) and, via (37), $\Psi_{\pm}$ are also known, the potentials are reconstructed in the usual way from the asymptotics of $\Psi_{\pm}$.

The third version of the inverse problem is a more traditional (nonlinear) Riemann-Hilbert (RH) problem. Solving the algebraic system $(17)_-$ with respect to $\boldsymbol{\varphi}$: $\boldsymbol{\varphi} = L(\boldsymbol{\psi}_-)$ (assuming local invertibility) and replacing this expression in the algebraic system $(17)_+$, one obtains the representation of the analytic eigenfunction $\boldsymbol{\psi}_+$ in terms of the analytic eigenfunction $\boldsymbol{\psi}_-$:

$$\boldsymbol{\psi}_+ = \boldsymbol{\mathcal{R}}(\boldsymbol{\psi}_-) = \boldsymbol{\psi}_- + \mathbf{R}(\boldsymbol{\psi}_-), \quad p \in \mathbb{R}, \tag{39}$$

which defines a *vector nonlinear RH problem on the real p axis*. The RH data $\boldsymbol{\mathcal{R}}$ are therefore constructed from the data $\boldsymbol{\mathcal{K}}$ by algebraic manipulation. Viceversa, given the RH data $\mathbf{R}$, one constructs the solutions $\boldsymbol{\psi}_{\pm}$ of the nonlinear RH problem (39) and, via the asymptotics (15), the potentials.

As for the other spectral data, one can show that the t-dependence of $\boldsymbol{\mathcal{R}}$ is described by (25) and the dKP constraint reads $\{\mathcal{R}_1, \mathcal{R}_2\}_{(\xi,p)} = 1$, while the reality constraint takes the form: $\boldsymbol{\mathcal{R}}(\overline{\boldsymbol{\mathcal{R}}(\bar{\boldsymbol{\xi}}, p)}, p) = \boldsymbol{\xi}$, $\forall \boldsymbol{\xi} \in \mathbb{R}^2$, for $p \in \mathbb{R}$.

Remark 2. Dressing schemes can be formulated from the three different inverse problems presented in this paper in a straightforward way.

Remark 3. The IST constructed in this paper allows one to solve the Cauchy problem for the whole hierarchy of PDEs arising from the commutativity equation $[\widehat{L}_1, \widehat{L}_2^{(n)}] = 0$, where the coefficients of the vector field $\widehat{L}_2^{(n)}$ are polynomials in p of arbitrary degree $n \in \mathbb{N}$.

Remark 4. There are deep similarities between the Cauchy problem for dKP and the Cauchy problem for the heavenly equation, recently solved in [6], since they are both based on the IST for Hamiltonian vector fields (the dKP equation is actually a geometric reduction of the heavenly equation [3]). There is, however, an important difference between these two cases. The vector fields of the dKP equation contain partial derivatives with respect to the spectral parameter p, unlike the case of the heavenly equation [6].

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