Effective action in the theory of quasi-ballistic disordered conductors

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We suggest an effective field theory for disordered conductors which describes the quantum kinetics of ballistically propagating electrons. This theory contains the nonlinear σ-model as its long wave limit. © *1995 American Institute of Physics.*

1. The nonlinear σ-model[1] has proven to be a useful tool in the description of various properties of disordered conductors. Any property, such as the conductivity, averaged over different time series of the random potential can be presented in this model as a statistical average with the free energy

$$F=\frac{\pi\nu}{8}\int d\mathbf{r}\ \mathrm{str}[D(\nabla Q)^2+2i\omega\Lambda Q],\quad Z=\int_{Q^2=1}\mathscr{D}Qe^{-F}. \tag{1}$$

The functional integral is taken over the 8×8 super-matrix $Q(\mathbf{r})$ subject to the constraint $Q^2=1$. Here and below we use the super-matrix version[1] of the nonlinear σ model.

This description is valid under the following two conditions:

i) the Fermi wavelength $\lambda_F=\hbar/p_F$ is much smaller than the mean free path l, i.e., $p_Fl/\hbar\gg1$;

ii) the typical wave vector q of the super-matrix fluctuations is smaller than $1/l$, i.e., $ql\ll1$.

These conditions mean that (i) the semi-classical description is applicable to electrons having the Fermi energy, and (ii) their motion is described by the diffusion equation. There are physical situations when condition (i) is fulfilled while condition (ii) is not, and the electrons propagate ballistically. This happens, for example, in a metal grain with diffuse boundary scattering if the bulk mean free path l is much larger than the grain size L, i.e., $l\gg L$.

In this letter we present a generalized version of the model (1), whose validity is no longer restricted by condition (ii). The generalized partition function correctly takes into account fluctuations with wave vectors $q\sim1/l$ and can therefore be used for the description of systems with ballistic electron motion.

We begin with a general expression for the free energy, which is obtained after averaging over the random potential, the Hubbard–Stratonovich decomposition of the quartic form, and integration over electronic degrees of freedom (see Ref. 1 for details and notations)

$$F=-\frac{1}{2}\,\mathrm{str}\,\ln[-i\hat{K}]+\frac{\pi\nu}{8\tau}\int \mathrm{str}\,Q^2(\mathbf{r})d\mathbf{r},\quad Z=\int \mathscr{D}Qe^{-F}, \tag{2}$$

$$\hat{K}=E-\hat{H}_0+\frac{\omega}{2}\Lambda+\frac{i}{2\tau}Q,\quad \hat{H}_0=\frac{(-i\hbar\nabla)^2}{2m}. \tag{3}$$

This expression appears at a preliminary stage in the derivation of Eq. (1), and the supermatrix Q is not yet restricted by the constraint $Q^2=1$.

Equation (2), in principle, could have served as the required generalization of the free energy (1). However, it is too detailed, being valid for super-matrices Q fluctuating with arbitrary wave vectors $\mathbf{q}$. It will be simplified in order to describe the small $\mathbf{q}$ fluctuations only ($q\ll p_F/\hbar$). The first step in the simplification is the same as in the derivation of the quantum kinetic equation in the Keldysh approach (see, for example, Ref. 2).

2. The Green function $G(\mathbf{r},\mathbf{r}'|Q)$ of the operator $\hat{K}$ obeys the equations

$$\left[E-\hat{H}_0(\mathbf{r})+\frac{\omega}{2}\Lambda+\frac{i}{2\tau}Q(\mathbf{r})\right]G(\mathbf{r},\mathbf{r}'|Q)=i\delta(\mathbf{r}-\mathbf{r}'), \tag{4}$$

$$[E-\hat{H}_0(\mathbf{r}')]G(\mathbf{r},\mathbf{r}'|Q)+G(\mathbf{r},\mathbf{r}'|Q)\left[\frac{\omega}{2}\Lambda+\frac{i}{2\tau}Q(\mathbf{r}')\right]=i\delta(\mathbf{r}-\mathbf{r}'). \tag{5}$$

Subtracting Eq. (5) from Eq. (4) and going to the Wigner representation

$$G(\mathbf{r},\mathbf{r}')=\int (d\mathbf{p})\tilde{G}\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{p}\right)e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \tag{6}$$

we can find, after integration over the modulus of the momentum $\mathbf{p}$, an equation for

$$g\mathbf{n}(\mathbf{r})=\frac{1}{\pi}\int d\xi\tilde{G}\left(\mathbf{r},\mathbf{n}\,\frac{\xi}{v_F}\right),\quad \mathbf{n}^2=1. \tag{7}$$

This equation can be written in the form

$$2v_F\mathbf{n}\frac{\partial g\mathbf{n}(\mathbf{r})}{\partial\mathbf{r}}=\left[i\omega\Lambda-\frac{Q}{\tau},g\mathbf{n}\right], \tag{8}$$

which resembles the quantum kinetic equation in the Eilenberger form.[3] The matrix $g\mathbf{n}(\mathbf{r})$ in this equation has the meaning of the distribution function at a coordinate $\mathbf{r}$ and momentum $\mathbf{p}=\mathbf{n}\cdot p_F$.

Being linear, Eq. (8) does not determine $g\mathbf{n}$ uniquely and must be supplemented with the normalization condition[2]

$$g_{\mathbf{n}}^2=1;\quad \mathrm{Tr}g\mathbf{n}=0. \tag{9}$$

The matrix $Q(\mathbf{r})$ is invariant with respect to charge conjugation,

$$\bar{Q}\equiv CQ^TC^T=Q, \tag{10}$$

where $\hat{C}$ is a certain matrix (see Ref. 1), for which $C^TC=1$. Taking the charge conjugate of Eq. (4) and using Eq. (10), we see that $\bar{G}(\mathbf{r},\mathbf{r}')$ obeys Eq. (5). Therefore,

$$\tilde{G}(\mathbf{r},\mathbf{r}')=G(\mathbf{r}',\mathbf{r}),\quad \tilde{\bar{G}}(\mathbf{r},\mathbf{p})=G(\mathbf{r},-\mathbf{p}),\quad \bar{g}\mathbf{n}(\mathbf{r})=g_{-}\mathbf{n}(\mathbf{r}). \tag{11}$$

Thus, Eq. (8) with the normalization condition (9) and the symmetries (11) is the long-wavelength limit in Eqs. (4) and (5). Our goal is to perform analogous simplification of the free energy (2).

3. An intermediate step is finding a functional Φ which reaches its extrema for solutions of Eq. (8). This equation resembles the equation of motion of a magnetic moment $\mathbf{M}$ in external magnetic field $\mathbf{B}$:

$$\frac{\partial \mathbf{M}}{\partial t}=[\mathbf{M}\times\mathbf{B}],\quad \mathbf{M}^2=1. \tag{12}$$

The action for this problem has the form (see, for instance, Ref. 4)

$$\mathscr{A}=\int_0^t dt'\mathbf{B}\mathbf{M}(t')+\int_0^1 dt'\int_0^1 du\tilde{\mathbf{M}}\cdot\left[\frac{\partial\tilde{\mathbf{M}}}{\partial t}\times\frac{\partial\tilde{\mathbf{M}}}{\partial u}\right], \tag{13}$$

where the function $\tilde{\mathbf{M}}(t,u)$ is introduced as

$$\tilde{\mathbf{M}}(t,0)=\mathbf{M}_0;\quad \tilde{\mathbf{M}}(t,1)=\mathbf{M}(t). \tag{14}$$

The second term in Eq. (13) does not depend upon the choice of $\mathbf{M}_0$ and values of $\tilde{\mathbf{M}}(t,u)$ for $0<u<1$, provided that $\mathbf{M}(0)=\mathbf{M}(t)$.

Following this analogy, we present Φ in the form

$$\Phi=\int d\mathbf{r}\ \mathrm{str}\left[\left(\frac{1}{\tau}Q(\mathbf{r})-i\omega\Lambda\right)\langle g(\mathbf{r})\rangle\right]+\frac{v_F}{2}\mathscr{W}\{g\mathbf{n}\}, \tag{15}$$

$$\langle g(\mathbf{r})\rangle=\int\frac{d\Omega\mathbf{n}}{4\pi}g\mathbf{n}(\mathbf{r}), \tag{16}$$

$$\mathscr{W}\{g\mathbf{n}\}=\int d\mathbf{r}\int\frac{d\Omega\mathbf{n}}{4\pi}\int_0^1 du\ \mathrm{str}\ \tilde{g}\mathbf{n}(\mathbf{r},u)\left[\frac{\partial\tilde{g}\mathbf{n}}{\partial u},\mathbf{n}\frac{\partial\tilde{g}\mathbf{n}}{\partial\mathbf{r}}\right], \tag{17}$$

$$\tilde{g}\mathbf{n}(\mathbf{r},0)=\Lambda;\quad \tilde{g}\mathbf{n}(\mathbf{r},1)=g\mathbf{n}(\mathbf{r}). \tag{18}$$

The functional derivative $\delta\Phi/\delta g\mathbf{n}$ must be taken with constraint (9), which guarantees that $g\mathbf{n}\delta g\mathbf{n}+\delta g\mathbf{n}g\mathbf{n}=0$, and an arbitrary variation $\delta g\mathbf{n}$ has the form $\delta g\mathbf{n}=[g\mathbf{n},a\mathbf{n}]$. As a result,

$$\delta\Phi=\int d\mathbf{r}\int\frac{d\Omega\mathbf{n}}{4\pi}\mathrm{str}\left(\left[\frac{1}{\tau}Q(\mathbf{r})-i\omega\Lambda,g\mathbf{n}\right]a\mathbf{n}\right)+\frac{v_F}{2}\delta\mathscr{W}, \tag{19}$$

where

$$\delta\mathscr{W}=4\int d\mathbf{r}\int\frac{d\Omega\mathbf{n}}{4\pi}\mathrm{str}\left(\mathbf{n}\frac{\partial g\mathbf{n}}{\partial\mathbf{r}}a\mathbf{n}\right). \tag{20}$$

Thus Eq. (15) gives the required functional.

4. Now we are prepared to show that in the limit $l\gg\lambda_F$ the partition function (2) reduces to the form

$$Z=\int_{g_{\mathbf{n}}^2=1}\mathscr{D}g\mathbf{n}(\mathbf{r})e^{-F}, \tag{21}$$

$$F=\frac{\pi\nu}{4}\left[\int d\mathbf{r}\ \mathrm{str}\left\{i\omega\Lambda\langle g(\mathbf{r})\rangle-\frac{1}{2\tau}\langle g(\mathbf{r})\rangle^2\right\}-\frac{v_F}{2}\cdot\mathscr{W}\{g\mathbf{n}\}\right], \tag{22}$$

$$\mathscr{W}\{g\mathbf{n}\}=\int d\mathbf{r}\int\frac{d\Omega\mathbf{n}}{4\pi}\int_0^1 du\ \mathrm{str}\ \tilde{g}\mathbf{n}(\mathbf{r},u)\left[\frac{\partial\tilde{g}\mathbf{n}}{\partial u},\mathbf{n}\frac{\partial\tilde{g}\mathbf{n}}{\partial\mathbf{r}}\right]. \tag{23}$$

Indeed, the following identity is valid:

$$Z_1\{Q\}\equiv\exp\left[\frac{1}{2}\mathrm{str}\ \ln(-i\hat{K})\right]=\int_{g_{\mathbf{n}}^2=1}\mathscr{D}g\mathbf{n}(\mathbf{r})\exp\left[\frac{\pi\nu}{4}\Phi\right]\equiv Z_2\{Q\}. \tag{24}$$

The free energy $\pi\nu\Phi/4$ in the partition function Z_2 has a deep minimum for $g\mathbf{n}(\mathbf{r})$ equal to $g_{\mathbf{n}}^{(0)}(\mathbf{r}|Q)$, which is the solution of Eq. (8). To saddle-point accuracy

$$\frac{\delta Z_2\{Q\}}{\delta Q(\mathbf{r})}=\frac{\pi\nu}{4\tau}\int\langle g(\mathbf{r})\rangle\exp\left[-\frac{\pi\nu}{4\tau}\Phi\right]\mathscr{D}g\mathbf{n}=\frac{\pi\nu}{4\tau}\langle g_{\mathbf{n}}^{(0)}\rangle\cdot Z_2\{Q\}. \tag{25}$$

On the other hand

$$\frac{\delta Z_1\{Q\}}{\delta Q(\mathbf{r})}=\frac{Z_1\{Q\}}{4\tau}G(\mathbf{r},\mathbf{r})=\frac{\pi\nu}{4\tau}\langle g_{\mathbf{n}}^{(0)}\rangle\cdot Z_1\{Q\}. \tag{26}$$

Thus the functionals $Z_{1,2}\{Q\}$ obey identical equations. Since $Z_1\{\Lambda\}=Z_2\{\Lambda\}=1$ the identity in (24) is proven.

Substituting Eq. (24) into Eq. (2) and taking the Gaussian integral over Q, we arrive at the final expression (21)–(23).

5. For small gradients, the free energy (22) reduces to the standard σ-model (1). To show this we expand the matrix $g\mathbf{n}$ into a sum over spherical harmonics $Y_{L,M}(\mathbf{n})$:

$$g\mathbf{n}(\mathbf{r})=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}g_{L,M}(\mathbf{r})\cdot Y_{L,M}(\mathbf{n})$$

and note that only the zeroth and first harmonics contribute to the functional integral (21):

$$g\mathbf{n}=Q(\mathbf{r})+\mathbf{J}(\mathbf{r})\cdot\mathbf{n}-\frac{Q\mathbf{J}^2}{6}. \tag{27}$$

The constraint $g^2=1$ now reads

$$Q^2=1,\quad Q\mathbf{J}+\mathbf{J}Q=0. \tag{28}$$

Substituting Eq. (27) into Eqs. (21)–(23) and using conditions (28), we obtain the partition function in the form

$$Z=\int\mathscr{D}Q\int\mathscr{D}\mathbf{J}e^{-F(Q,\mathbf{J})},\quad F(Q,\mathbf{J})=\frac{\pi\nu}{4}\int d\mathbf{r}\ \mathrm{str}\left\{i\omega\Lambda Q+\frac{\mathbf{J}^2}{6\tau}-\frac{v_F}{3}(\nabla Q)Q\mathbf{J}\right\}. \tag{29}$$

After doing the Gaussian integration over $\mathbf{J}$ in Eq. (29) we arrive, finally, at Eq. (1).

6. Equations (21)–(23) can be generalized to describe ballistic motion in the presence of external fields. In the general case the electron is described by the classical Hamiltonian $H(p_i, x_i)$ and the kinetic equation (8) has the form (see Ref. 2):

$$\{H(x,p),g(x,p)\}=\left[\left(\frac{i\omega\Lambda}{2}-\frac{Q}{2\tau}\right),g(p,x)\right], \tag{30}$$

where $\{H,g\}$ denotes the Poisson brackets

$$\{H(x,p),g(x,p)\}=\frac{\partial H}{\partial p_i}\frac{\partial g}{\partial x_i}-\frac{\partial H}{\partial x_i}\frac{\partial g}{\partial p_i}.$$

Equation (30) is still a first-order differential equation, and the generalization of expression (22) for the free energy has the form

$$F=\frac{\pi}{4}\int dx_i dp_i \delta(E-H(p,x))\,\mathrm{str}\left\{i\omega\Lambda g-\frac{g\langle g\rangle}{2\tau}-\frac{1}{2}\int_0^1 du\tilde{g}(x,p,u)\left[\frac{\partial\tilde{g}}{\partial u},\{H,\tilde{g}\}\right]\right\}, \tag{31}$$

where

$$\langle g(x)\rangle=\frac{1}{\nu}\int dp_i'\,\delta(E-H(p',x))g(x,p').$$

7. As an application of Eq. (31), let us consider the derivation of the Pruisken action[5] for a two-dimensional electron gas in a perpendicular magnetic field B. To simplify the treatment, we consider only the case of a classically weak field:

$$\Omega_c\tau\ll 1; \quad \Omega_c=\frac{eB}{mc}, \tag{32}$$

when there is no Landau quantization and the density of states ν is a constant. Nevertheless, we take into account that the presence of a magnetic field reduces the symmetry of the g matrix, and that g belongs to a unitary ensemble. The Poisson brackets in magnetic field are

$$\{H,g\}=v_F\mathbf{n}\frac{\partial g\mathbf{n}}{\partial\mathbf{r}}+\Omega_c\left[\mathbf{n}\times\frac{\partial g\mathbf{n}}{\partial\mathbf{n}}\right] \tag{33}$$

and the free energy (31) has the following form

$$F=\frac{\pi\nu}{4}\int d\mathbf{r}\ \mathrm{str}\left\{i\omega\Lambda\langle g\rangle-\frac{\langle g\rangle^2}{2\tau}-\frac{1}{2}\int_0^1 du\left\langle\tilde{g}(x,p,u)\left[\frac{\partial\tilde{g}}{\partial u},v_F\mathbf{n}\frac{\partial\tilde{g}}{\partial\mathbf{r}}+\Omega_c\left[\mathbf{n}\times\frac{\partial\tilde{g}}{\partial\mathbf{n}}\right]\right]\right\rangle\right\}. \tag{34}$$

In the diffusive limit the expansion (27) can be used, which leads to the following expression for the free energy as a functional of Q and $\mathbf{J}$;

$$Z=\int \mathscr{D}Q\int \mathscr{D}\mathbf{J}e^{-F(Q,\mathbf{J})},$$

$$F(Q,\mathbf{J})=\frac{\pi\nu}{4}\int d\mathbf{r}\ \mathrm{str}\left\{i\omega\Lambda Q+\frac{\mathbf{J}^2}{4\tau}-\frac{v_F}{2}(\nabla Q)Q\mathbf{J}-\frac{\Omega_c}{2}Q[\mathbf{J}\times\mathbf{J}]\right\}. \tag{35}$$

The last term in the free energy (35) does not vanish because the components of the matrix $\mathbf{J}$ do not commute. Under conditions (32), the Gaussian integration over $\mathbf{J}$ may be performed, with the vector product in Eq. (35) as a perturbation, to yield, finally, the free energy in the form

$$F=\frac{\pi}{8e^2}\int d\mathbf{r}\ \mathrm{str}(\sigma_{xx}(\nabla Q)^2+2\sigma_{xy}Q[\nabla_x Q,\nabla_y Q]), \tag{36}$$

where

$$\sigma_{xx}=e^2\nu D,\quad \sigma_{xy}=\sigma_{xx}\Omega_c\tau. \tag{37}$$

8. There is a topological question, related to the $\mathscr{W}$ term in the free energy (22): is it always possible to construct the functional $\mathscr{W}\{g\}$ whose variation is given by Eq. (20)? The prescription (17) gives the $\mathscr{W}$ term for the functions $g(\mathbf{r})$, which are close to $g_0(\mathbf{r})\equiv\Lambda$. The question is whether such a functional can be defined globally.

The answer depends upon the topology of the constant energy surface $H(\mathbf{r},\mathbf{p})=\mathrm{E}$ in the phase space $\{x_i,p_i\}$. For the cases of billiards and space dimension $d>1$ the functional $\mathscr{W}$ does exit.

For a one-dimensional system, $\mathscr{W}$ can only be found as a multivalued functional, just as the action (13). This causes no trouble, provided $\pi\hbar\nu v_F$ is an integer. This integer exactly equal to the waveguide channel number in the wire.

An accurate mathematical formulation and the proof of these statements will be presented elsewhere.

9. So far, we have considered only systems with a finite amount of disorder. One can see, however, that expression (31) remains meaningful even as $\tau\rightarrow\infty$. Therefore, we expect that the free energy $F_\infty=F(\tau\rightarrow\infty)$ describes a clean system with the Hamiltonian H. As a consequence, the partition function $Z_\infty=\int\mathscr{D}g\exp(-F_\infty)$ with the proper source terms gives the level statistics.

In the low-frequency limit ($\omega\rightarrow 0$) only the zero-mode $g^0(r,p)$, such that $\{H,g^0\}=0$, contributes to Z_∞. There are two possibilities:

i) the Hamiltonian system under consideration is integrable and there exists a set of integrals of the motion $\{I_1,\ldots I_n\}$, $\{H,I_k\}=0$. Under this condition the energy levels are characterized by the eigenvalues of $\{I_1,\ldots I_n\}$ and do not repel each other. Therefore the level statistics is Poissonian;

ii) the classical dynamics is chaotic and the only integral of the motion is energy. In this case the zero-mode is constant in the phase space and Z_∞ reduces to the form

$$Z_\infty=\int_{g^2=1}\mathscr{D}g\exp\left(-\frac{\pi\nu\omega}{4}\mathrm{str}(\Lambda g)\right) \tag{38}$$

which leads to the Wigner–Dyson (WD) level statistics.[1]

In the chaotic case deviations from the WD statistics occur for frequencies larger than the inverse time of flight through the system. These deviations are described by the small fluctuations of g about several stationary points Λ_i, similar to what has been recently shown by Andreev and Altshuler (AA) for diffusive systems.[7] In complete agreement with the general AA-conjecture, the deviation from the WD statistics is described by the determinant of some operator. It follows from our discussion that this is the Liouvillean operator

$$\hat{L}=\frac{\partial H}{\partial p}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial p}.$$

10. In conclusion, we would like to emphasize that the theory presented here contains the diffusive σ model as a limiting case and provides it with a physically motivated regularization of the infinities at short distances.

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[1] K. B. Efetov, Adv. Phys. **53**, 250 (1984).
[2] A. Schmid in *Nonequilibrium Superconductivity, Phonons and Kapitza Boundaries,* edited by K. E. Gray, Plenum Press, New York (1981), p. 423.
[3] G. Eilenberger, Z. Phys. **214**, 195 (1968).
[4] E. S. Fradkin, *Field Theories of Condensed Matter Systems,* Addison–Wesley (1991).
[5] A. M. M. Pruisken, Nucl. Phys. B **235**, 277 (1984).
[6] B. L. Altshuler and B. I. Shklovskiĭ, Zh. Éksp. Teor. Fiz. **91**, 220 (1986) [Soviet Phys. JETP **64**, 127 (1986)].
[7] A. Andreev and B. L. Altshuler, Phys. Rev. Lett. (1995), submitted.

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