The level spacing statistics in a finite 1D disordered sample

I. V. Kolokolov
INFN, Sez. di Milano, via Celoria 16, 20133 Milano, Italy;
Budker Institute of Nuclear Physics, 630090 Novoskbirsk, Russia

(Submitted 11 April 1994; resubmitted 1 June 1994)
Pis'ma Zh. Eksp. Teor. Fiz. **59**, No. 12, 841–844 (25 June 1994)

The distribution function $\mathcal{P}(\Delta)$ of the spacing Δ between nearest energy levels is calculated for a one-dimensional disordered sample with a finite length L. The evaluation proceeds in terms of the Schrödinger equation with a random potential, rather than random matrix ensembles. The common case in which the wavelength of a particle is small compared with the mean free path is considered. Thus Δ is expressed in terms of a solution of the equation with a given energy and all the moments $\langle \Delta^m \rangle$ and then the $\mathcal{P}(\Delta)$ are calculated with use of a recently developed functional integral method for a 1D random potential problem.

Statistical properties of the level spacing Δ in random quantum systems have been the subject of extensive investigation of the pioneering studies.[1,2] They are also the focus of attention of recent papers.[3] On the other hand, the results of numerical experiments of chaotic quantum systems[4–6] can be interpreted in terms of quasi-one-dimensional quantum mechanics with a random Hamiltonian.[7] The quasi-one-dimensional behavior is shown to be equivalent in many cases to the one in the strictly 1D random potential problem with some effective parameters.[8–11]

The statistics of Δ in an essentially disordered case has been studied analytically in the thermodynamic limit only. In the case of a finite sample, however, it is of interest to study the physics of mesoscopic systems and the quantum dynamics in a finite-dimensional Hilbert space.[6,12] In addition, the probability of finding small Δ is determined completely by finite-size effects (see the discussion below).

In the present letter we calculate the distribution function $\mathcal{P}(\Delta)$ for a Schrödinger particle situated in the finite 1D interval $(-L,L)$. The potential $U(x)$ in the particle Hamiltonian $\hat{\mathcal{H}}=-d^2/dx^2+U(x)$ is assumed to be a random function of x, which obeys the white-noise Gaussian statistics: $\langle U(x)U(x')\rangle = D\,\delta(x-x')$. The result is obtained in the "fast-phase" limit $kL \gg 1$, $kl \gg 1$, where k is the particle's momentum, and $l=4k^2/D$ is the localization length. The relationship between l and L is arbitrary.

In this study we used the results and notation of Ref. 13, where the new functional integral approach to the 1D random potential problem was developed.

The eigenfunction $\psi(x)$ of $\hat{\mathcal{H}}$ is the solution of the equation $(\hat{\mathcal{H}}-k^2)\psi(x)=0$ which obeys certain conditions at the points $x=-L$ and $x=L$, e.g., $\psi(-L)=\psi(L)=0$. Let us consider the solution $u_k(x)$ of the Caushy problem $(\hat{\mathcal{H}}-k^2)u_k(x)=0$, $u_k(-L)=0$, $u_k'(-L)$-1. If we represent $u_k(x)$ as $a(x)\sin\,\phi_k(x)$, then in fast-phase limit mentioned above the level spacing is

$$\Delta=\frac{2\pi k}{|\partial_k\phi_k(L)|}. \tag{1}$$

(There is no summation over k in this equation.) Excluding the free motion, we see that the phase $\phi_k(L)$ can be written as $\phi_k(L)=2kL+\Phi_s(L/l)$, where the contribution $\Phi_s(L/l)$ is due to the potential and depends on the parameters of the problem via the ratio L/l only. This term and its derivative with respect to k are not small by themselves. However, the next derivatives of Φ_s with respect to k have the additional factor $1/kl$ compared with $\partial_k\Phi_s$ and can be ignored. Requiring the variation of the phase between two nearest levels to be equal to 2π, we obtain Eq. (1). With the same precision it leads to the expression of Δ in terms of $u_k(x)$:

$$\Delta=2\pi\frac{(u_k'^2+k^2u_k^2}{u_k\partial_k u_k'-u_k'\partial_k u_k}\Big|_{x=L}=\frac{2\pi k v_1(L)v_2(L)}{\int_{-L}^{L}v_1(y)v_2(y)dy}. \tag{2}$$

Here $u_k'\equiv\partial_k u_k$ and the "plane wave components" $v_{2,1}(x)=e^{\pm ikx}[u_k'(x)\pm iku_k(x)]$ are introduced. The formalism developed in Ref. 13 allows us to represent the moments $\langle\Delta^m\rangle$, $m\geqslant 1$ as quantum mechanical matrix elements:

$$\langle\Delta^m\rangle=\left(\frac{\pi kl}{2}\right)^m\frac{1}{\Gamma(m)}\langle e^{\xi/2}|e^{-2L\hat{\mathscr{H}}}|e^{-(m+1/2)\xi}\rangle, \tag{3}$$

where ξ is the coordinate of the 1D quantum mechanics, and $\hat{H}$ has the form

$$\hat{H}=-\frac{1}{l}\partial_\xi^2+\frac{1}{4l}e^{-\xi}+\frac{1}{4l}. \tag{4}$$

The brackets $\langle...|...|...\rangle$ on the right-hand side of (3) and below denote usual scalar product: $\langle f_1(\xi)|\hat{A}|f_2(\xi)\rangle=\int_{-\infty}^{+\infty}d\xi f_1(\xi)\hat{A}f_2(\xi)$, where $f_{1,2}(\xi)$ are some functions, and $\hat{A}$ is an operator. From a given set of moments we can restore immediately the Laplace transform $P(s)$ of the distribution function $\mathscr{P}(\Delta)$. Using the integral representation (Eq. 8.315 in Ref. 14) of $1/\Gamma(m)$, we obtain the expression

$$P(s)=\sum_{m=0}^{\infty}\frac{(-s)^m}{m!}\langle\Delta^m\rangle=1+\langle e^{\xi/2}|e^{-2L\hat{H}}|\Upsilon(\xi;s)\rangle, \tag{5}$$

where

$$\Upsilon(\xi;s)=\frac{e}{2\pi}e^{-\xi/2}\int_{-\infty}^{+\infty}dte^{it}\exp\left(-\frac{\pi ks}{2l(1+it)}e^{-\xi}\right).$$

The matrix element in (5) can be evaluated noting that

$$\frac{1}{l}e^{\xi/2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu}\cosh\pi\nu[K_{2i\nu+1}(le^{-\xi/2})-K_{2i\nu-1}(le^{-\xi/2})], \tag{6}$$

and

$$\hat{H}K_{2i\nu\pm1}(le^{-\xi/2})=\frac{1}{l}(\nu^2\mp i\nu)K_{2i\nu\pm1}(le^{-\xi/2}). \tag{7}$$

The last equation means that (6) represents the function $e^{\xi/2}$ as a linear combination of the eigenfunctions of $\hat{H}$.[1] Substituting (6) and (7) into (5) and performing the inverse Laplace transform, we obtain the following expression after some manipulations:

$$\mathcal{P}(\Delta)=\frac{l}{k\pi^3}\sqrt{\frac{\Delta l}{8k}}\int_{-\infty}^{+\infty}d\tau\cosh\tau\exp\left(-\frac{\Delta l}{2\pi k}\cosh^2\tau\right)$$

$$\times\int_{-\infty}^{+\infty}d\nu\,\frac{\sin(2\nu L/l)}{\nu}\cosh\pi\nu\exp\left(-2\frac{L}{l}\nu^2+2i\nu\tau\right). \tag{8}$$

In deriving (8) we used the integral representation (Eq. 8.432 in Ref. 14) of the function $K_\mu(z)$. In the limit $L\to\infty$ for a given Δ expression (8) reduces to the well-known Poisson distribution:[15]

$$\mathcal{P}(\Delta)=\frac{l}{2k\pi}\exp\left(-\frac{\Delta l}{2k\pi}\right). \tag{9}$$

The finite-size corrections to (9) are of the order of magnitude $\sim\exp(-L/2l)$. As $\Delta\to0$ and $L\sim l$, the asymptotic limit of the function (8) is

$$\mathcal{P}(\Delta)\approx\frac{l}{8k}\sqrt{\frac{l}{2\pi L}}\exp\left[-\frac{l}{8L}\left(\ln\frac{8\pi k}{\Delta l}-\frac{2L}{l}\right)^2\right]\mathcal{F}\left(\frac{l}{2L}\ln\frac{8\pi k}{\Delta l}\right), \tag{10}$$

where the function $\mathcal{F}(x)$ is

$$\mathcal{F}(x)=\sqrt{\frac{1}{x}}\exp\left(x(1-\ln x)-\frac{l}{8L}(\ln x-1)^2\right). \tag{11}$$

Thus, if $\Delta\to0$, the distribution function $\mathcal{P}(\Delta)$ goes to zero faster than any power of Δ and cannot be described rigorously by the Wigner distribution with any set of parameters. This point differs from the results of numerical simulations of chaotic quantum systems[6] and it could be a consequence of the topology of the boundary conditions. The logarithmically normal distribution (10) does not correspond, however, to any self-averaging quantity. The large Δ tail coincides with the function (9).

The representation (1) becomes exact in the small scattering limit. Thus, the final expression (8) must reproduce, in the limit $l\to\infty$, the equidistant level structure. Changing the integration variable $\nu\to\nu l/L$, we reduce $d\nu d\tau$-integration to the saddle point ($\tau=i\pi/2$, $\nu=\pm\pi/4$) contribution. The latter gives $\mathcal{P}(\Delta)=\delta(\Delta-\pi k/L)$.

It is worth noting that the expectation value of the inverse level spacing, $\langle1/\Delta\rangle$, calculated by means of the distribution (8) is not affected by the localization effects

$$\langle\Delta^{-1}\rangle=\frac{L}{\pi k} \tag{12}$$

for an arbitrary l/L.

I wish to thank B. V. Chirikov for stimulating advice. I also thank D. Shepelyansky for discussions.

[1)]The functions on each side of (6) are not normalizable and they cannot be considered an expansion over a basis in the Hilbert space.

[1] F. J. Dyson, J. Math. Phys. **3**, 140 (1962). E. P. Wigner, Proc. Cambridge Philos. Soc. **47**, 790 (1951).
[2] L. P. Gor'kov and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. **48**, 1407 (1965) [JETP **21**, 940 (1965)].
[3] A. G. Aronov, V. E. Kravtsov, and I. V. Lerner, JETP Lett. **59**, No. 1, 39 (1994).
[4] B. V. Chirikov, Phys. Rept. **52**, 263 (1979).
[5] G. Casati, B. V. Chirikov, I. Guarnery, and D. Shepelyansky, Phys. Rept. **154**, 77 (1987).
[6] F. M. Izrailev, Phys. Rept. **196**, 299 (1990).
[7] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **71**, 412 (1993); A. D. Mirlin and Y. V. Fyodorov, J. Phys. A **26**, L551 (1993).
[8] K. B. Efetov, Adv. in Phys. **32**, 53 (1983).
[9] A. A. Abrikosov and I. A. Ryzhkin, Adv. in Phys. **27**, 146 (1978).
[10] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67**, 2405 (1991).
[11] I. V. Kolokolov, Preprint IFUM 456/FT, Milano, 1994.
[12] G. Casati, B. V. Chirikov, I. Guarnery, and F. M. Israilev, Phys. Rev. E **48**, R1613 (1993).
[13] I. V. Kolokolov, JETP **76**, 1099 (1993).
[14] I. S. Gradstein and I. M. Ryzhyk, *Tables of Integrals, Series and Products.* Academic (N.Y.), 1980.
[15] V. M. Malkin, Preprint Inst. of Nucl. Physics, Novosibirsk, 1981.

Published in English in the original Russian journal. Reproduced here with stylistic changes by the Translations Editor.