L-A pair of a system of coupled equations of the gravitational and electromagnetic fields

V. A. Belinskiĭ
L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted 15 May 1979)
Pis'ma Zh. Eksp. Teor. Fiz. **30**, No. 1, 32–35 (5 July 1979)

The *L-A* pair that corresponds to a coupled system of Einstein–Maxwell equations for the case when the metrics and electromagnetic potential depend on only two variables and the electromagnetic field invariant $F^{ik}F_{ik}$ equals zero has been found.

PACS numbers: 04.40. + c

In an earlier work co-authored with Zakharov,[1] the author presented the *L-A* equations that correspond to the gravitational equations in a vacuum for a case when the metrics depend only on time t and one spatial variable z. This technique, it was noted,[2] is also applicable in a space filled with an ideal fluid with the equation of state $\epsilon = p$. In this paper, we would like to point out one more case of integrable equations of the general theory of relativity. It appears that the coupled system of the Einstein-Maxwell equations postulates the existence of the *L-A* pair if (1) the metrics and the electromagnetic potential depend on only two variables, (2) charges and currents are nonexistent, and (3) the electromagnetic field invariant $F_{ik}F^{ik}$ equals zero.

We shall describe the interval in the same form as in Ref. 1

$$-ds^2 = f(-dt^2 + dz^2) + g_{ab}\, dx^a dx^b, \tag{1}$$

where f and g_{ab} depend only on t and z. Let us denote the coordinates as $(x^0,x^1,x^2,x^3) = (t,x,y,z)$. The Latin indices a, b, c, and d shall pertain to coordinates x and y and shall range through numbers 1 and 2. The three Latin indices i, k, and l refer to the four-dimensional space and we shall assign to them values 0, 1, 2, and 3. Let the potential A_i and tensor $F_{ik} = A_{k,i} - A_{i,k}$ correspond to the electromagnetic field. In this case, the Einstein-Maxwell equations are as follows:

$$F_{ik}F^{ik} = 0, \quad F^{ik}_{\ \ ;k} = 0, \quad R^k_i = \frac{1}{2} F_{il}F^{kl} \tag{2}$$

The conditions of consistency of this system with the metrics [Eq. (1)] and the gauge invariance of the potential A_i indicate that the electromagnetic field should be as follows:

$$A_0 = A_3 = 0, \quad A_a = A_a(t, z), \quad F_{0a} = \dot{A}_a, \quad F_{3a} = A'_a, \tag{3}$$

where the dot and prime denote derivatives with respect to t and z. Having introduced the light variables ζ and η

$$t = \zeta - \eta, \quad z = \zeta + \eta \tag{4}$$

it can easily be shown that Eq. (2) takes the following form:

$$g^{ab}A_{a,\zeta}A_{b,\eta}=0\,; \tag{5}$$

$$(\alpha g^{ab}A_{b,\eta})_{,\zeta}+(\alpha g^{ab}A_{b,\zeta})_{,\eta}=0, \tag{6}$$

$$(\alpha g_{ac,\zeta}g^{cb})_{,\eta}+(\alpha g_{ac,\eta}g^{cb})_{,\zeta}=-\alpha g^{cb}(A_{a,\zeta}A_{c,\eta}+A_{a,\eta}A_{c,\zeta}), \tag{7}$$

$$(\ln f)_{,\zeta}(\ln\alpha)_{,\zeta}-(\ln\alpha)_{,\zeta\zeta}-\frac{1}{4}g_{ac,\zeta}g_{bd,\zeta}g^{cb}g^{ad}=\frac{1}{2}g^{ab}A_{a,\zeta}A_{b,\zeta} \tag{8}$$

$$(\ln f)_{,\eta}(\ln\alpha)_{,\eta}-(\ln\alpha)_{,\eta\eta}-\frac{1}{4}g_{ac,\eta}g_{bd,\eta}g^{cb}g^{ad}$$

$$\frac{1}{2}g^{ab}A_{a,\eta}A_{b,\eta}, \tag{9}$$

where α denotes the square root of a determinant of a two-dimensional matrix g_{ab} which satisfies, as Eqs. (5) and (7) indicate, a simple wave equation

$$\det g_{ab}=\alpha^2,\quad \alpha_{,\zeta\eta}=0. \tag{10}$$

The fundamental equations in the system [Eqs. (5)–(9)] are the first three, i.e., 5–7, which determine the components of the metric tensor g_{ab} (and its inverse g^{ab}) and electromagnetic potential A_a. Equation (5) constitutes an additional condition $F_{ik}F^{ik}=0$, Eq. (6) is Maxwell's equation and Eq. (7) follows from the *ab*-terms of Einstein's equations. Upon solution of this system, the metric coefficient f is found from Eqs. (8) and (9).

To construct the *L-A* equations that correspond to Eqs. (5)–(7), we shall introduce the same operators D_1 and D_2 as in Ref. 1

$$D_1=\partial_\zeta-\frac{2\alpha_{,\zeta}\lambda}{\lambda-\alpha}\partial_\lambda\,,\qquad D_2=\partial_\eta+\frac{2\alpha_{,\eta}\lambda}{\lambda+\alpha}\partial_\lambda \tag{11}$$

and a ψ-function which should now consist of the two-dimensional matrix $\psi_{ab}(\lambda,\zeta,\eta)$ and a two-dimensional vector $\psi_a(\lambda,\zeta,\eta)$. The required linear differential equations, which determine ψ_{ab} and ψ_a, have the following form:

$$D_1\psi_{ab}=\frac{1}{\lambda-\alpha}(-\alpha g_{ad,\zeta}g^{ac}-\alpha A_aA_{d,\zeta}g^{dc})\psi_{cb}$$
$$+\frac{1}{\lambda-\alpha}(\alpha g_{ad,\zeta}g^{dc}A_c+\alpha A_aA_{d,\zeta}g^{dc}A_c-\alpha A_{a,\zeta})\psi_b, \tag{12}$$

$$D_2\psi_{ab} = \frac{1}{\lambda+\alpha}(\alpha g_{ad,\eta} g^{dc} + \alpha A_a A_{d,\eta} g^{dc})\psi_{cb}$$

$$+\frac{1}{\lambda+\alpha}(-\alpha g_{ad,\eta} g^{dc} A_c - \alpha A_a A_{d,\eta} g^{dc} A_c + \alpha A_{a,\eta}), \qquad (13)$$

$$D_1\psi_a = \frac{1}{\lambda-\alpha}(-\alpha A_{d,\zeta} g^{dc})\psi_{ca} + \frac{1}{\lambda-\alpha}(\alpha A_{d,\zeta} g^{dc} A_c)\psi_a, \qquad (14)$$

$$D_2\psi_a = \frac{1}{\lambda+\alpha}(\alpha A_{d,\eta} g^{dc})\psi_{ca} + \frac{1}{\lambda+\alpha}(-\alpha A_{d,\eta} g^{dc} A_c)\psi_a . \qquad (15)$$

Evidently, for $A_0 = 0$, the above equations yield the L-A equations for a vacuum given in Ref. 1.

Direct verification shows that the compatibility conditions for Eqs. (12)–(15) fully coincide with Eqs. (5)–(7) and produce no new relations with respect to g_{ab} and A_a.

If a solution of Eqs. (12)–(15) exists, the metric coefficients g_{ab} and electromagnetic potential A_a are directly given by the values of the ψ-functions for a zero value of the spectral parameter λ in accordance with the following relations:

$$\psi_{ab}(0,\zeta,\eta) = g_{ab} + A_a A_b, \quad \psi_a(0,\zeta,\eta) = A_a . \qquad (16)$$

We shall now show a method of determining Eqs. (12)–(15). We shall combine the terms g_{ab} and A_a into a symmetric three-dimensional matrix j as follows:

$$j = \begin{pmatrix} g_{ab} + A_a A_b & A_a \\ A_a & 1 \end{pmatrix} \qquad j^{-1} = \begin{pmatrix} g^{ab} & -g^{ac} A_c \\ -g^{ac} A_c & 1 + g^{dc} A_d A_c \end{pmatrix}. \qquad (17)$$

The determinants of j and g_{ab} are coincident and equal to α^2. If we now construct two three-dimensional matrices U and V

$$U = -\alpha j_{,\zeta} j^{-1}, \qquad V = \alpha j_{,\eta} j^{-1} \qquad (18)$$

it can readily be shown that one 3×3 matrix equation

$$U_{,\eta} - V_{,\zeta} = 0 \qquad (19)$$

is equivalent to the system of Eqs. (5)–(7). Equations (18) and (19), together with the relation $\det j = \alpha^2$, where α is the same function as in Eq. (10), are reiterations of a system of equations studied in Ref. 1, except for the fact that their constituent matrices and one of the diagonal components of the matrix j should be a unity ($j_{33} = 1$). Thus, the L-A-pair for Eqs. (18) and (19) is expressed as follows:

$$D_1\psi = \frac{1}{\lambda-\alpha} U\psi, \quad D_2\psi = \frac{1}{\lambda+\alpha} V\psi, \qquad (20)$$

where ψ is a three-dimensional matrix. The system of Eqs. (12) –(15) follows directly from the above, if ψ_{ab} is considered to be the left upper two-dimensional bloc of the three-dimensional matrix ψ, and ψ_a, a two-dimensional vector of its third row ($\psi_a = \psi_{3a}$). Evidently, as pointed out earlier, this part of Eq. (20) constitutes the necessary and sufficient *L-A* equations for the initial problem expressed in Eqs. (5) –(7).

[1]V.A. Belinskiĭ and V.E. Zakharov, Zh. Eksp. Teor. Fiz. **75**, 1953 (1978). [Sic].
[2]V.A. Belinskiĭ, *ibid.* **77**, No. 10 (1979) [Sov. Phys. JETP (1979)].