New class of field theories

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It is shown that the field theories, which are equivalent to classical gases with generalized charges having certain discrete symmetries, are integrable.

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In statistical physics an important role is played by two-dimensional lattice models of magnets, which in the classical case are described by the Hamiltonian

$$H^{(n)} = \pm \frac{1}{2} \Sigma J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j , \quad (1)$$

where $\mathbf{s}_i$ are unit vectors specified in the lattice points, which can take on values in the entire n-dimensional sphere [continuous symmetry group $O(n)$][1] or at certain points (discrete symmetry groups).[2] The sign $\pm$ corresponds to the antiferro- (AF) and ferro- (F) magnets, respectively (for diminishing J_{ij}).

The quadratic form of Eq. (1) in $\mathbf{s}_i$ allow us to express the statistical sum $Z^{(n)}$ in terms of the variables coupled with $\mathbf{s}_i$. Using the new variables $Z^{(n)}$ has the form of a generating functional of the field theory for the lattice. For example, for the Ising AF model ($n = 1$)

$$Z^{(1)} = \sum_{\{s_1\}} \exp(-\beta H^{(1)}) = \int D\phi_i \exp\{-\frac{1}{2} \Sigma \phi_i J_{ij}^{-1} \phi_j + \Sigma V(\sqrt{\beta}\, \phi_i)\},$$

where $V(x) = \ln \cos x$. (2)

Conversion to the Ising F model corresponds to the substitution $\sqrt{\beta} \to \pm i\sqrt{\beta}$, i.e., Laplace transformation replaces Fourier transformation. As a result, $Z^{(1)}$ remains the same, except $V(x) = \ln \operatorname{ch} x$.[3]

On the other hand, it is known that the field theories with $V(x) = \cos x$ and $V(x) = \operatorname{ch} x$ are integrable,[4] and the first one is equivalent to the Coulomb gas[5] [transformation for gases is similar to the transformation (2),[5,6]] whose Hamiltonian has the form of Eq. (1) with $n = 1$. Therefore, in going over to a gas with the Hamiltonians from (1) with $J_{ij} = -1/2\pi \ln|\mathbf{x}_i - \mathbf{x}_j|$ (the spins $\mathbf{s}_i$ play the role of generalized charges), we obtain local field theories with V that do not contain the logarithm.[7] As a result, we obtain the field theories with the following potentials for the gases with the

Hamiltonian $H^{(n)}(n\geqslant 2)$:

$$V_n(x) = (x/2)^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(x) \qquad \text{for } F \text{ gas,} \tag{3a}$$

$$V_n(x) = (x/2)^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(x) \qquad \text{for } AF \text{ gas.} \tag{3b}$$

where $x = \sqrt{\beta}\,|\vec{\phi}|$, and $|\vec{\phi}| = [\Sigma_1^n \phi_i^2]^{1/2}$. We note that for the F gas with $n=0$ (Newton gas) we obtain the Liouville field theory.[8]

A question arises concerning the properties of these theories. Some of them for the theories with V_n from Eq. (3b) are given in Ref. 7. It is not clear at present whether these theories are integrable, because at $n>1$ their symmetry is continuous. It would be desirable, therefore, to examine first the field theories that are equivalent to the gases with discrete groups.

We shall begin with the F gases. The potential V in the general case has the following form

$$V(\vec{\phi}) = \sum_{i=1}^{N} \exp(\mathbf{s}^{(i)} \cdot \vec{\phi}'), \quad \sum_{i=1}^{N} \mathbf{s}^{(i)} = 0, \quad \vec{\phi}' = \sqrt{\beta}\,\vec{\phi}, \tag{4}$$

where $\Sigma_{i=1}^{n}$ is the sum over all positions of the spin $\mathbf{s}^{(i)}$ in the sphere. We write the potentials for certain simple, nontrivial groups.

1) $n=2$, Z_n symmetry groups.

$$N=3, \quad V(\vec{\phi}) = e^{\phi'_2} + 2e^{-\frac{1}{2}\phi'_2}\operatorname{ch}\frac{\sqrt{3}}{2}\phi'_1,$$

$$N=4, \quad V(\vec{\phi}') = 2[\operatorname{ch}\phi'_1 + \operatorname{ch}\phi'_2].$$

2) $n=3$, symmetry groups—regular polyhedron groups. We shall label them according to the number of vertices N.

$N=4$, tetrahedron group, see general equation (5).

$N=6$, octahedron group, $V(\vec{\phi}^{\,1}) = 2\Sigma_1^3 \operatorname{ch}\phi_i^1$.

3) n arbitrary. The order of the first nontrivial group is $n+1$, to which corresponds V from Eq. (4), where

$$\mathbf{s}^{(i)} = (a_n,\ b_n a_{n-1},\ b_n b_{n-1} a_{n-2}, \ldots,\ b_n \ldots b_i,\ 0, \ldots), \quad i = 1, 2, \ldots n,$$
$$\mathbf{s}^{(n+1)} = (1, 0, \ldots), \quad a_n = -1/n, \quad b_n^2 = 1 - a_n^2. \tag{5}$$

In writing V for groups of $(n+1)$ order, we always used a coordinate system of the following type. The $\mathbf{e}_n$ axis is directed along $\mathbf{s}^{(n+1)}$. After projection on the $(n-1)$ dimension space, we obtain a $(n-2)$ dimensional sphere with radius b_n. This sphere contains vectors corresponding to the projections of the remaining n vectors of the original figure. The same procedure is used for them. The next group has the order $2n$, and $V(\vec{\phi}'_i) = 2\Sigma_1^n \text{ch}\, \phi'_i$ corresponds to it. In this case the $\mathbf{s}^{(i)}$ vectors coincide with the coordinate axes. We note that the lattice models with interaction of the nearest neighbors, which have the aforementioned discrete symmetries of the $(n+1)$-type groups, are self-dual. This is a consequence of the presence of only one excitation energy.[2,9]

It follows from V for the $2n$-order group that these field theories are integrable. It is very likely that the field theories corresponding to the $(n+1)$-order groups are also integrable (due to their distinction). We shall demonstrate this for $n+1=3$ (Z_3 group) by reduction to the "two-dimensional Toda chain." The latter is a "two-dimensional" generalization of the regular Toda chain,[10] which is also integrable.[11] Its Lagrangian is

$$L_T = \frac{1}{2}(\partial_\mu x_i)^2 - V_T(x_{i+1} - x_i), \quad V_T(x_i) = \sum_1^M \exp(x_i), \quad x_{i+M} = x_i. \tag{6}$$

Let $M=3$. In the normal coordinates[12] Eq. (6) can be rewritten as follows:

$$L_T = \frac{1}{2}(\partial_\mu q_i)^2 - V(q_i), \quad V(q_i) = e^{-\sqrt{2}q_2} + 2\,\text{ch}\left(\sqrt{\frac{3}{2}}\, q_1\right) e^{\frac{\sqrt{2}}{2} q_2}. \tag{7}$$

If we assume that the "center of gravity" is $q_3 = (1/\sqrt{3})(\Sigma_1^3 x_i) = 0$, $-\sqrt{2}q_2 = \phi'_2$, and $\sqrt{2}q_1 = \phi'_1$, then Eq. (7) becomes the Lagrangian of the field theory corresponding to the Z_3 group with a doubled coupling constant. Unfortunately, the field theory for the groups with $n+1>3$ could not be converted into the Toda chain with $M = n+1$. The possibility of their integrability is being studied. The two-dimensional Toda chain with arbitrary M is equivalent to a certain "gas" field theory. Specifically, $M=4$ corresponds to the "isotropic"-type potential of the Ashkin-Teller model, which is also self-dual in the lattice. It is conceivable that each Toda chain corresponds to a self-dual model in the lattice.

We shall discuss briefly the theories corresponding to the AF gases with discrete groups. The real V are obtained only for the groups with an inversion center. The simpler, nontrivial groups have the order $2n$ with the vectors lying on the axes. They correspond to the integrable potentials $V = 2\,\Sigma_1^n \cos \phi'_i$. The question concerning the integrability of the remaining V is still open.

At the same time, the properties of the AF gases with a continuous symmetry[7] stated in Ref. 7 (with a slight modification for the groups without the inversion center) are transferable to the AF gases with a discrete symmetry, since they can have $\mathbf{s}^{(i)}$ vectors of the same length, and the gas, as a whole, is neutral.

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