

Thermodynamics of a 2D electronic gas in a strong magnetic field

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A method for calculating the statistical sum of a 2D electronic gas in a strong magnetic field is proposed. An explicit expression for the quantities describing the exchange effects at a temperature $T \rightarrow 0$ is derived for the interparticle-interaction potentials which differ only slightly along the magnetic length.

In the wake of the discovery of the fractional quantum-mechanical Hall effect,¹ many papers attempting to explain this heretofore puzzling phenomenon have been published. Without mentioning all the procedures which make use of the familiar method of the Laughlin trial functions,² we will point out the statistical-sum method which was examined in Refs. 3 and 4. On the other hand, Kivelson *et al.*⁵ have suggested a fundamentally new approach to the fractional quantum-mechanical Hall effect. Using the path-integration method, Kivelson *et al.*⁵ have studied the question as to whether the free energy, a function of the particle density, has singularities attributable to the exchange effects. Using a method which differs from that of Ref. 5, we have obtained results which are at variance with the results of Ref. 5. The method which we describe below is based on a direct calculation of the statistical sum of the interacting particles in a strong magnetic field H .

The problem consists of finding the statistical sum

$$Z = \sum_n e^{-\beta E_n}, \quad \beta = 1/T, \quad (1)$$

where E_n are the eigenvalues of the energy of the interacting particles at the lowest Landau level. We assume that the particles do not move to other levels as a result of the interaction. (The scale energy of the binary interaction is considerably lower than $\hbar\omega_c$, where ω_c is the cyclotron frequency). In this case, the right side of (1) is

$$\sum_n \left\{ 1 - \beta V_{nn} + \frac{\beta^2}{2!} \sum_m V_{nm} V_{mn} \dots \right\}, \quad (2)$$

where V is the potential energy of the interaction of particles, and the states n and m correspond to the lowest Landau level. In the vector-potential gauge $\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{r}$, the corresponding normalized single-particle wave functions are

$$\psi_m(z) = (2\pi 2^m m!)^{-1/2} z^m e^{-|z|^2/4}, \quad (3)$$

where $z = x + iy$, and the magnetic length $l_H = (\cos \hbar/eH)^{1/2}$ is assumed to be unity.

Projection of the matrix elements of the interaction onto the lowest Landau level gives rise to the following explicit form of the arbitrary term of the series in (2):

$$\frac{\beta^l}{l!} (-1)^l \int \prod_{i=1}^N d^2 z_i \prod_{j=1}^N d^2 z_{1j} \dots \prod_{k=1}^N d^2 z_{lk} V(\mathbf{z}_1) V(\mathbf{z}_2) \dots V(\mathbf{z}_l) \times g(\mathbf{z}, \mathbf{z}_1) g(\mathbf{z}_1, \mathbf{z}_2) \dots g(\mathbf{z}_{l-1}, \mathbf{z}_l) g^{(\mathcal{A})}(\mathbf{z}_l, \mathbf{z}'). \quad (4)$$

Here $d^2 z = dx dy$, $\mathbf{z} = \{z_1, \dots, z_N\}$, N is the number of particles, $V(\mathbf{z}) = \frac{1}{2} \sum_{i \neq j} U(|z_i - z_j|)$,

$$g(\mathbf{z}, \mathbf{z}') = \sum_n \Psi_n(\mathbf{z}) \Psi_n^*(\mathbf{z}'), \quad g^{(\mathcal{A})}(\mathbf{z}, \mathbf{z}') = \sum_n \Psi_n^{(\mathcal{A})}(\mathbf{z}) \Psi_n^{(\mathcal{A})*}(\mathbf{z}'). \quad (5)$$

In (5) the functions $\Psi_n(\mathbf{z})$ correspond to a set of orthonormalized N -particle wave functions at the lowest Landau level, in which the Pauli principle is ignored, whereas the functions $\Psi_n^{(\mathcal{A})}(\mathbf{z})$ are antisymmetric functions relative to the permutation of particles. Only the function $g^{(\mathcal{A})}(\mathbf{z}_l, \mathbf{z})$ need be introduced in expression (4), since the matrix elements of the interaction are nonvanishing only between the wave functions with identical permutation symmetry.

Using expression (3), we easily see that

$$g(\mathbf{z}_1, \mathbf{z}_2) = \frac{1}{(2\pi)^N} \exp\left(\frac{1}{2} \mathbf{z}_1 \mathbf{z}_2^* - \frac{1}{4} |\mathbf{z}_1|^2 - \frac{1}{4} |\mathbf{z}_2|^2\right), \quad (6)$$

where

$$\mathbf{z}_1 \mathbf{z}_2^* = \sum_{i=1}^N z_{1i} z_{2i}^*, \quad \text{and } |\mathbf{z}|^2 = \sum_{i=1}^N |z_i|^2.$$

The functions $g(\mathbf{z}_1, \mathbf{z}_2)$ and $g^{(\mathcal{A})}(\mathbf{z}_1, \mathbf{z}_2)$ are related by a rather obvious relation

$$g^{(\mathcal{A})}(\mathbf{z}_1, \mathbf{z}_2) = \frac{1}{N!} \sum_P \text{sign}(P) g(\mathbf{z}_1, P\mathbf{z}_2). \quad (7)$$

Here P is an arbitrary permutation of variables $\{z_i\}$, and $\text{sign}(P)$ is its sign.

The product of the g functions in (4) is given by

$$\sum_P \text{sign}(P) \exp\left\{-\frac{1}{4} |\mathbf{z} - \mathbf{z}_1|^2 - \frac{1}{4} |\mathbf{z}_1 - \mathbf{z}_2|^2 - \dots - \frac{1}{4} |\mathbf{z}_l - P\mathbf{z}|^2 - \frac{i}{2} \sum_{i=1}^N [\mathbf{r}_i \times \mathbf{r}_{1i} + \mathbf{r}_{1i} \times \mathbf{r}_{2i} + \dots + \mathbf{r}_{li} \times P\mathbf{r}_i] \cdot \mathbf{n}\right\}, \quad (8)$$

where \mathbf{r}_j is a two-dimensional radius vector of the j -th particle, which is perpendicular to $\mathbf{n} = \mathbf{H}/H$. Since all characteristic differences $|z_j - z_m| \sim 1$, we can set $V(\mathbf{z}_1) = V(\mathbf{z}_2) = \dots = V(\mathbf{z}_l)$ for the potential in (4) which varies slowly over an interval on the order of l_H (the effective range of the potential is a characteristic integration interval, $a \gg l_H$). For this class of potentials, simple calculations put the statistical sum in the form

$$Z = \frac{1}{N!} \sum_P \text{sign}(P) \int \prod_{i=1}^N \frac{d^2 z_i}{2\pi} \exp \left[\frac{1}{2} \mathbf{z}(P\mathbf{z})^* - \frac{|\mathbf{z}|^2}{2} - \beta V(\mathbf{z}) \right]. \quad (9)$$

The expression inside the integral sign depends solely on the class of permutations of the symmetrical group. We know⁶ that the class of permutations is determined by the set of permutation cycles $\{\alpha\}$ such that

$$\alpha_1 + 2\alpha_2 + \dots + N\alpha_N = N, \quad \text{sign}(P) = (-1)^{\alpha_2 + \alpha_4 + \dots}, \quad (10)$$

where α_n is the number of cycles of length n in $\{\alpha\}$. The number of permutations in the class $\{\alpha\}$ is

$$N! (\alpha_1! 2^{\alpha_2} \alpha_2! \dots N^{\alpha_N} \alpha_N!)^{-1}. \quad (11)$$

Taking (10) and (11) into account, we can write expression (9) in the form

$$Z = \sum_{\{\alpha\}} \frac{(-1)^{\alpha_2 + \alpha_4 + \dots}}{\alpha_1! 2^{\alpha_2} \alpha_2! \dots} \int \prod_{i=1}^N \frac{d^2 z_i}{2\pi} \exp \left[\frac{1}{2} \mathbf{z}(P_{\{\alpha\}} \mathbf{z})^* - \frac{|\mathbf{z}|^2}{2} - \beta V(\mathbf{z}) \right]. \quad (12)$$

The cycle $(1\dots m)$ inside the sign of the exponential function in (12) corresponds to the expression

$$-\frac{1}{4}(\mathbf{r}_1 - \mathbf{r}_2)^2 - \frac{1}{4}(\mathbf{r}_2 - \mathbf{r}_3)^2 - \dots - \frac{1}{4}(\mathbf{r}_m - \mathbf{r}_1)^2 - \frac{i}{2}[\mathbf{r}_1 \times \mathbf{r}_2 + \dots + \mathbf{r}_m \times \mathbf{r}_1] \mathbf{n}. \quad (13)$$

The imaginary part in (13) describes the commensurability (see Ref. 4). In the case of free particles, all thermodynamic quantities can easily be found by using the large canonical ensemble.⁷

We will explain in conclusion how a transition to the results of Ref. 5 can be made. At very low temperatures, we assume that the integration in (12) is determined by the factor $e^{-\beta V}$ and that

$$e^{-\beta V} \simeq e^{-\beta E_0} \rho(z_1, \dots, z_N), \quad (14)$$

where E_0 is the energy of a classical Wigner crystal, and the function ρ corresponds to the arrangement of particles in the lattice sites (the δ -shaped function). As a result of integration, the points \mathbf{r}_i in (13) are situated in the lattice sites. In the limit $\nu \rightarrow 1$ (ν is the degree of filling of the Landau level) the value of β must increase progressively in order to satisfy the approximation in (14), while at $\nu = 1$ this approximation does not apply, since the system is liquid in this case (see also Ref. 8). In the summation over the nearest sites, each cycle of length m corresponds to the factor

$$\exp[-\alpha m + i\pi + i\pi(\nu^{-1} - 1)N_A], \quad (15)$$

where the first term in square brackets describes the overlap of the wave functions, and the remaining terms correspond to the imaginary part in (13), with allowance for the permutation sign. In (15) we have $N_A = \pm(\nu S/\pi l_H^2)$, where S is the area included in the permutation cycle (the sign of its circuit is taken into account). For a hexagonal lattice we have $\alpha = (\pi/\nu\sqrt{3})$.

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