“Irreversibility” of the flux of the renormalization group in a 2D field theory

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There exists a function $c(g)$ of the coupling constant $g$ in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where $c$ is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

The renormalization group is one of the most powerful methods for qualitative studies in field theory.\textsuperscript{1,2} The procedure for determining the renormalization group can be roughly summarized as follows:\textsuperscript{1} We denote by $S(g,a)$ an action functional of a (Euclidean) field theory which is an integral of the local density, $S = \int \sigma(g,a,x)dx$, equipped with an ultraviolet cutoff $a$ and depending on a (possibly infinite) set of dimensionless parameters $g = (g^1,g^2,...)$, which are known as “coupling constants.” A basic assumption is that there exists a single-parameter group of motions in the space $(Q)$ of coupling constants $g, R, Q \to Q$, of such a nature that a field theory describable by an action $S(R,g, e^t a)$ is equivalent to the original theory with the action $s(g,a)$ in the sense that all the correlation functions of the two theories agree at scales $x > e^t a (t > 0)$. The components of the vector field which generate the renormalization group are called “$\beta$ functions”:

$$\frac{d}{dt} g^i = \beta^i(g) \frac{d}{dt}.$$

(1)

Some of the information on the ultraviolet behavior of the field theory is lost under renormalization transformations with $t > 0$, since in the field theory it is not legitimate to examine correlations at scales smaller than the cutoff. We would therefore expect that a motion of the space $Q$ under the influence of the renormalization group would become an “irreversible” process, similar to the time evolution of dissipative systems. In the present letter we restrict the discussion to a 2D field theory, and for this case we establish the following general properties of the renormalization group:

1. There exists a function $c(g) > 0$ of such a nature that we have

$$\frac{d}{dt} c \equiv \beta^i(g) \frac{\partial}{\partial g^i} c(g) \leq 0$$

(2)

(a repeated index implies a summation). The equality in (2) is reached only at fixed points of the renormalization group, i.e., at $g = g_\ast$ [$\beta^i(g_\ast) = 0$].

2. The fixed points (here and below, we mean “critical” fixed points, at which the correlation radius is infinite\textsuperscript{1}) are stationary for $c(g)$; i.e., we have $\beta^i(g) = 0 \to \partial c / \partial g^i = 0$. At the critical fixed points, the 2D field theory has an infinite conformal
symmetry. The corresponding generators $L_n, \, n = 0, \pm 1, \pm 2, \ldots,$ form a Virasora algebra

$$[ L_n, \, L_m ] = (n - m)L_{n + m} + \frac{\tilde{c}}{12}(n^3 - n)\delta_{n + m, 0}, \quad (3)$$

where the numerical parameter $\tilde{c}$ (the "central charge") is an important characteristic of a conformal field theory. It generally takes on different values for different fixed points; i.e., $\tilde{c} = \tilde{c}(g_\ast)$.

3. The value of $c(g)$ at the fixed point $g_\ast$ is the same as the corresponding central charge in (3); i.e., $c(g_\ast) = \tilde{c}(g_\ast)$.

The proof of this "c-theorem" is based on the conditions of renormalizability, positivity, the translational and rotational symmetries of the field theory, and certain special properties of a 2D conformal field theory. Spatial symmetries in a local field theory lead to the existence of a local energy-momentum tensor $T_{\mu\nu}(x) = T_{\nu\mu}(x)$ which satisfies the equation $\partial_{\mu}T_{\mu\nu} = 0$. We introduce the complex coordinates $(z, \bar{z}) = (x^1 + ix^2, x^1 - ix^2)$, and we use the notation $T = T_{zz}$, and $\Theta = T_{z\bar{z}}$. We also define the scalar local fields

$$\Phi_i(g, x) = \frac{\partial}{\partial g^i} \phi(g, a, x). \quad (4)$$

The exact meaning of the assertion that a field theory is renormalizable is that for all $g$ the field $\Theta$ can be expanded in basis (4):

$$\Theta = \beta^i(g)\Phi_i, \quad (5)$$

where the coefficients $\beta^i(g)$ are the same as in (1). We define the functions

$$C(g) = 2\alpha^4 \langle T(x)T(0) \rangle \big|_{x^2 = x_0^2}; \quad (6a)$$

$$H_i(g) = \alpha^2 x^2 \langle T(x)\Phi_i(0) \rangle \big|_{x^2 = x_0^2}; \quad (6b)$$

$$G_{ij}(g) = \alpha^4 \langle \Phi_i(x)\Phi_j(0) \rangle \big|_{x^2 = x_0^2}; \quad (6c)$$

where $x_0 \gg a$ is an arbitrary scale ("normalization point"). At this point we set $x_0 = 1$. By virtue of the positivity condition in the field theory, the symmetric matrix $G_{ij}(g)$ is positive definite and may be thought of as the metric in $Q$. Combining the requirement $\partial_{\mu}T_{\mu\nu} = 0$ with (5) and with the Callan-Simanchik equation, we find the relations

$$\frac{1}{2} \beta^i \partial_i C = -3\beta^i H_i + \beta^k \beta^j \partial_k H_i + \beta^{ik} (\partial_k \beta^i) H_i; \quad (7a)$$

$$\beta^{ik} \partial_k H_i + (\partial_i \beta^{ik}) H_k - H_i = -2\beta^{ik} G_{ik} + \beta^j \beta^{ik} \partial_k G_{ij} + \beta^j (\partial_i \beta^{ik}) G_{jk} + \beta^j (\partial_j \beta^{ik}) G_{ik}, \quad (7b)$$

where $\partial_i = \partial / \partial g^i$. In deriving (7) we made use of the following expression for the
matrix \( \gamma_i^j(g) \):

\[
\gamma_i^j(g) \Phi_j \equiv \left( \frac{1}{2} a \frac{\partial}{\partial a} - \beta^k \frac{\partial}{\partial g^k} \right) \Phi_i = (\partial_i \beta^j) \Phi_j
\]  \hspace{1cm} (8)

For the function

\[
c(g) = C(g) + 4\beta^i H_i - 6\beta^i \beta^j G_{ij}
\]  \hspace{1cm} (9)

we find from (7)

\[
\beta^i \partial_i c = -12\beta^i \beta^j G_{ij}.
\]  \hspace{1cm} (10)

directly verifying Assertion 1. Assertion 3 follows from (9) and from the definition of the central charge \( \tilde{c}(g_\ast) \) as the numerical coefficient in the correlation function \( \langle T(z) T(0) \rangle g_\ast = z^{-\tilde{c}(g_\ast)}/2 \). To prove Assertion 2, we consider the critical fixed point \( g_\ast \), and we choose a coordinate system in \( Q \) such that we have \( g_\ast = 0 \) and

\[
G_{ij}(g) = \delta_{ij} + 0(g^2).
\]  \hspace{1cm} (11)

In this case the vectors \( \Phi_i(g_\ast, \tau) \) are conformal fields and have certain anomalous dimensionalities \( d_i \). Near the point \( g_\ast = 0 \), the function \( c(g) \) can be calculated by perturbation theory; the result is

\[
c(g) = \tilde{c}(g_\ast) - 6\varepsilon_i g_i g^i + 2C_{ijk} g_i g^j g^k + O(g^4),
\]  \hspace{1cm} (12)

where \( 2\varepsilon_i = 2 - d_i \). Assertion 2, in particular, follows from (12). We also note that in the special case of "soft" perturbations, with \( |\varepsilon_i| \ll 1 \), it can be shown that the coefficients \( C_{ijk} \) in (12) are the same as the structure constants of the operator algebra of a conformal field theory, \( g_\ast \). For the \( \beta^i \) functions we find

\[
\beta^i(g) = \varepsilon_i g^i - \frac{1}{2} C_{ijk} g_j g^k + O(g^3).
\]  \hspace{1cm} (13)

(No summation is to be carried out in the first term.) At the specified accuracy, therefore, the following relation holds near the fixed point:

\[
\beta^i(g) = -\frac{1}{12} G^{ij}(g) \frac{\partial}{\partial g^j} c(g),
\]  \hspace{1cm} (14)

where \( G^{ik} G_{kj} = \delta^i_j \).

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5J. Glimm and A. Jaffe, Mathematical Methods of Quantum Field Theory (Russ. transl., Mir, Moscow, 1984).

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