# Reduction in integrable systems. The reduction group

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By studying infinite reduction groups we are led to a generalization of the inverse-problem method to the case in which the spectral parameter $\lambda$ lies on an arbitrary algebraic curve. We outline the classification of integrable models and their reductions and construct a new nontrivial example of an integrable model involving the tetrahedral group.

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A large number of nonlinear field models which are integrable by the inverse-problem method are now known. Their classification is an important problem which, in our view, should be solved in terms of the inverse-problem method. In essence, the problem reduces to describing the possible "*L-A* pairs" and enumerating the reductions in each "gauge class". In previous papers [1–3] the author has shown that description of the reductions is intimately connected with the study of the particular symmetry group of the consistency conditions and, hence, of the integrable models themselves (the reduction group). In this letter we show, after a short introduction (Section 1), that studying the infinite reduction groups leads to interesting generalizations of the inverse-problem method. We outline the classification scheme of the integrable models and, in Section 3, give an example of a new integrable field model (12).

1. We shall examine a pair of linear equations for the matrix function $\psi$

$$L_1\psi = \psi_\xi - U_1 \psi = 0, \quad L_2 \psi = \psi_\eta - U_2\psi = 0, \tag{1}$$

where $U_{1,2}$ are $N\times N$ matrix functions of the coordinates $\xi$ and $\eta$ and the complex parameter $\lambda$. Their consistency condition

$$U_{1\eta} - U_{2\xi} + [U_1\ U_2] = 0 \quad (\text{for all } \lambda) \tag{2}$$

leads to a nonlinear system of equations (see Refs. 2, 4, and 5) which is amenable to study by the inverse-problem method.

We denote by $\Psi(\lambda)$ the set of fundamental solutions of the pair of equations (1) for a fixed value of $\lambda$. We require that the set $\Psi(\lambda)$ be operated on by a symmetry group constructed in the following way. Let $G$ be a group of $N\times N$ matrices which in general depend on $\xi$, $\eta$, and $\lambda$, and let $\mathscr{G}$ be a discrete group of linear fractional transformations of the $\lambda$ plane. We examine a subgroup $G_R$ of the direct product $\hat{G}\times\mathscr{G}$, i.e. $G_R$ consists of certain ordered pairs of the form $(\hat{g}, g(\lambda))$, where $g\in\hat{G}$, $g(\lambda)\in\mathscr{G}$. The group $G_R$ operates on the functions $\psi(\lambda)$ in the following way:

$$\psi(\lambda) \to \psi_1(\lambda) = \hat{g}\,\psi(g(\lambda)); \ \psi(\lambda) \in \Psi(\lambda), \ (\hat{g}, g(\lambda)) \in G_R, \tag{3}$$

  

The requirement of symmetry with respect to $G_R$ is contained in the condition that $\psi_1(\lambda)\in\Psi(\lambda)$. For this it is obvious that

$$\hat{g}^{-1} L_{1,2}(\lambda)\hat{g} = L_{1,2}(g(\lambda)) \tag{4}$$

and, hence, restrictions are imposed on the form of the "potentials" $U_{1,2}$:

$$\hat{g}^{-1} U_1(\lambda)\hat{g} - \hat{g}^{-1}\hat{g}_\xi = U_1(g(\lambda)),\ \hat{g}^{-1}U_2(\lambda)\hat{g} - \hat{g}^{-1}\hat{g}_\eta = U_2(g(\lambda)). \tag{5}$$

By direct substitution one is convinced that relations (5) are compatible with the equations of motion (2) for any $\hat{g}$ and $g(\lambda)$. We note that the group $G_R$ can be enlarged by transformations of the form

$$\psi(\lambda) \to \hat{t}(\psi^u(t(\lambda)))^{-1}\in\psi(\lambda),\ \psi(\lambda) \to \hat{r}\,\bar{\psi}(r(\bar{\lambda}))\in\psi(\lambda) \tag{6}$$

$$\psi(\lambda') \to \hat{h}(\psi^+(h(\bar{\lambda})))^{-1}\in\psi(\lambda);\ t(\lambda),\ r(\bar{\lambda}),\ h(\bar{\lambda})\in\mathcal{G},\ \hat{t},\ \hat{r},\ \hat{h}\in\hat{G},$$

which also lead to restrictions on the form of $U_{1,2}$ which are compatible with (2) (see Refs. 2 and 3).

In other words, by requiring symmetry in the problem of the consistency of Eq. (1) with respect to the group $G_R$ we construct in a systematic way the reduction of system (2) to a system with substantially fewer fields but possessing hidden symmetry. This symmetry is manifested, for example, in the action-angle variables, where it leads to transformations in the space of the "fields" and to complex transformations of the parameter $\lambda$. We note that in relativistically invariant problems the parameter $\lambda$ sometimes has the meaning $e^{\beta}$ (where $\beta$ is the speed), and there is an analogous interpretation in the quantum-mechanical version of the inverse-problem method.[6] The connection of the group $G_R$ with the group of gauge transformations and several examples are described in detail in Ref. 2. In what follows we shall assume for simplicity that the matrices $\hat{g}\in\hat{G}$ do not depend on $\xi$, $\eta$, $\lambda$.

2. We shall also examine transformations from $G_R$ of the form $(I,\tilde{g}(\lambda))$, where $\tilde{g}(\lambda)$ form a normal subgroup $\mathcal{G}^1\in\mathcal{G}$. Equations (5) imply that the matrices $U_{1,2}(\lambda)$ are automorphic functions of the group $\mathcal{G}^1$, and it is therefore sufficient to define them in the fundamental region $\Gamma^1 = C/\mathcal{G}^1$. The group $\mathcal{G}^1$ identifies the boundary of the region $\Gamma^1$ and converts it to a Riemann surface, which we shall also denote by $\Gamma^1$.

A natural generalization of the rational consistency problem[4,5] is the problem in which the matrix functions $U_{1,2}$ are meromorphic on the Riemann surface $\Gamma^1$. The poles of these functions lie on the orbits $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_e\}$ of the finite group $\mathcal{G}_1 = \mathcal{G}/\mathcal{G}^1$. We recall that the orbit $A_1$ or point $a_1$ is what we call the set of points $g(a_1)$, where $g(\lambda)\in\mathcal{G}_1$. We expand the matrices $U_{1,2}(\xi, \eta, \lambda)$

$$U_{1,2}(\xi, \eta, \lambda) = \sum_{k=1}^{M_{1,2}} U^k_{1,2}(\xi, \eta)\, f_k^{1,2}(\lambda) + U^{\circ}_{1,2}(\xi, \eta) \tag{7}$$

in the basis functions $f_k^{1,2}(\lambda)$, which are meromorphic on $\Gamma^1$ and have no singularities except for poles at $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_e\}$, respectively. Under operation by $\mathcal{G}_1$ the functions $f_k^{1,2}(\lambda)$ transform as follows:

$$f_k^{1,2}(g(\lambda)) = \sum_{S=1}^{M_{1,2}} F_{ks}^{1,2}(g) f_s^{1,2}(\lambda). \tag{8}$$

The matrices $F_{ks}^{1,2}(g)$ in an obvious way specify the representation of the group $\mathcal{G}_1$, which depends on the orbits $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_e\}$ and on the choice of basis $f_k^{1,2}$. When Eqs. (7) and (8) are taken into acount, relations (5) assume the form

$$\hat{g}^{-1} U_{1,2}^k \hat{g} = \sum_{s=1}^{M_{1,2}} F_{ks}^{1,2}(g) U_{1,2}^s, \quad \hat{g}^{-1} U_{1,2}^{\circ} \hat{g} = U_{1,2}^{\circ}. \tag{9}$$

We examine transformations of the group $G_R$ which are of the form $(g^\nu, I)$, where $g^\nu$ form a normal subgroup $\hat{G}^1$ of group $\hat{G}$. They reduce to similarity transformations without a change in $\lambda$ $(F_{k,s}^{1,2}(I) = \delta_{ks})$. Such transformations will be called transformations of the first kind:

$$\hat{g}^{-1} U_{1,2}^k \hat{g} = U_{1,2}^k \; ; \quad k \neq 0, \; \hat{g} \in \hat{G}^1. \tag{10}$$

If $\hat{G}^1 = I$ or lies at the center of an unexpected representation of the group $\hat{G}$, then Eq. (10) does not impose any restrictions on the form $U_{1,2}^k$. The case of an unexpected $\hat{G}^1$ corresponds to the trivial reduction $U_{1,2}^k \equiv 0$. The factor group $\hat{G}_1 = \hat{G}/\hat{G}^1$ connects matrices $U_{1,2}^k$ of different $k$ (transformations of the second kind). We shall call reductions in which there are no transformations of the first kind "canonical" reductions.

Thus, the analytical structure of the matrices $U_{1,2}$ is determined by a discrete group $\mathcal{G}$, its subgroup $\mathcal{G}^1$, and the set of poles $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_e\}$. The matrix structure is classified by the groups $\hat{G}$, $\hat{G}^1$, and their representations.

3. The finite groups of linear fractional transformations of the complex $\lambda$ plane are exhausted by the groups $Z_n$, $D_N$, and the groups of the regular polyhedra (tetrahedron, octahedron, icosahedron).[7] Examples of reductions due to $Z_N$ and $D_N$ were examined in Refs. 2 and 3. We shall examine the tetrahedral group $T$, which is generated by two transformations, $Q$ and $J$. Transformation $Q$ is a rotation of the vertex by $2\pi/3$, and $J$ is a rotation by $\pi$ about an axis passing through the centers of opposite edges. By inscribing a tetrahedron in a Riemann sphere (see the stereographic projection in Fig. 1), one can easily construct a representation of the group $T$ by linear fractional transformations. Let the group $\hat{G}$ be isomorphic to $T$, and $\hat{G}_R$ be the diagonal of $\hat{G} \times \mathcal{G}$, i.e., be generated by the elements $(Q,q)$ and $(J,j)$. We choose matrix representations of $\hat{G}$ and $\mathcal{G}$ in the following form $(\omega = \exp(2\pi i/3))$

$$j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad q = \frac{1}{2} \begin{pmatrix} i+1 & i\sqrt{2} \\ i\sqrt{2} & 1-i \end{pmatrix}, \quad Q = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J = -\frac{1}{3} \begin{pmatrix} 1 & -2i & 2 \\ 2i & 1 & -2i \\ 2 & 2i & 1 \end{pmatrix}$$

It is clear that $\mathcal{G}^1 = \hat{G}^1 = I$ and, hence, there are no conditions of the first kind (10).

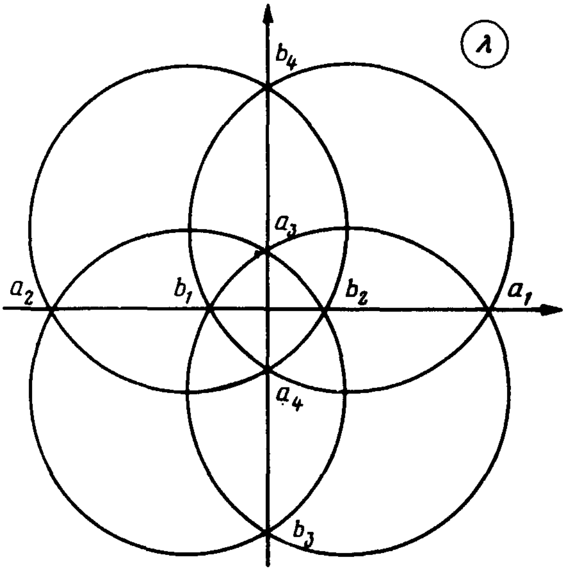


FIG. 1.

We examine two degenerate orbits $A$ and $B$, corresponding to the vertices of the tetrahedron $\{a_1,\ldots,a_4\}$ and the centers of opposite faces $\{b_1,\ldots,b_4\}$ (see Fig. 1.). We take as bases the rational functions

$$f_k^1 = \frac{\lambda - b_k}{\lambda - a_k}, \quad f_k^2 = \frac{\lambda - a_k}{\lambda - b_k}; \quad k = 1, \ldots 4.$$

The matrix $F^{1,2}$ is easily evaluated:

$$F^{1,2}(q) = \begin{bmatrix} \omega^{\pm 1} & 0 & 0 & 0 \\ 0 & 0 & -\omega^{\pm 1}(2\pm\sqrt{3}) & 0 \\ 0 & 0 & 0 & \omega^{\pm 1} \\ 0 & -\omega^{\pm 1}(2\pm\sqrt{3}) & 0 & 0 \end{bmatrix}, \quad F^{1,2}(j) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(the upper sign corresponds to $F^1$, the lower to $F^2$). It follows rather easily from Eq. (9) that $U^0_{1,2} = 0$ (owing to the irreducibility of the representation of group $T$, while the other four matrices $U^k_{1,2}$ can be expressed linearly in terms of $U^1_{1,2}$, where

$$U_1^1 = \begin{pmatrix} 0 & u_1 & 0 \\ 0 & 0 & u_2 \\ u_3 & 0 & 0 \end{pmatrix}, \qquad U_2^1 = \begin{pmatrix} 0 & 0 & v_3 \\ v_1 & 0 & 0 \\ 0 & v_2 & 0 \end{pmatrix} \tag{11}$$

Conditions (2) is equivalent to the equality to zero of all the residues at the points

$a_i$, $b_i$. Inasmuch as the residues are connected by transformations from the reduction group, it is sufficient to see them equal to zero at the points $a_1$ and $b_1$:

$$u_{1\eta} + \frac{2}{3}(1+\sqrt{3})(v_3 - iv_2)v_1 = 0, \quad v_{1\xi} + \frac{2}{3}(1-\sqrt{3})(u_3 - iu_2)v_1 = 0$$

( + cyclic permutations of indices 1, 2, 3). (12)

It is easy to see that $\partial_\eta(u_1u_2u_3) = 0$ and $\partial_\xi(v_1v_2v_3) = 0$, and, hence, there are only four independent equations for the four complex functions. By supplementing the group $T$ with reflections and the group $G_R$ with transformations of the form (6), we will arrive at four equations for four real functions.

The initial system with four poles each operator led to 162 equations for 180 real functions. Actually, we found a nontrivial *ansatz* in this nonlinear system and reduced it to four first-order equations.

Let $\mathscr{G}$ be the group of displacements $\lambda \to \lambda + m_1\omega_1 + m_2\omega_2$, $(\mathrm{Im}(\omega_1/\omega_2) \neq 0)$, and $\hat{G}$ be an arbitrary finite group with two generatrices $\hat{g}_1$ and $\hat{g}_2 (\hat{g}_1^{k_1} = \hat{g}_2^{k_2} = I)$. The fundamental region $\Gamma^1$ is equivalent to a torus and is contained in a parallelogram with side $(k_1\omega_1, k_2\omega_2)$. If $U_{1,2}$ have simple, noncoincident poles, can be expanded in Weierstrass $\zeta$ functions.

$$U_\alpha(\lambda, \xi, \eta) = \sum_{e,m} \zeta(\lambda - a_\alpha - l\omega_1 - m\omega_2)U_\alpha^{lm}(\xi, \eta) \quad (\alpha = 1, 2),$$

where the sumation is over the lattice contained in $\Gamma^1$. The group $G^1$ coincides with the commutator $[\hat{G}, \hat{G}]$ and, hence, $\hat{G}_1$ is Abelian. If the representation of $\hat{G}$ is irreducible, the condition that the sum of the residues be equal to zero $(\Sigma^{l,m} U_\alpha^{lm} \approx 0)$ is automatically satisfied, and a nontrivial (noncentral) commutator will lead to conditions of the first kind. The important example of an "elliptical $L$-$A$ pair" was found by Borovik and Skyanin.[8]

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