Singularities of the energy “gap” of a superconductor

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The existence of singularities in the energy “gap” $\Delta(p,\omega)$ in $(p,\omega)$ space is predicted. It is shown that in the general case the singularities in $\Delta(p,\omega)$ are due to a change in the topology of the line of intersection of the Fermi surface with the phonon equal-energy surfaces.

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It is known that the energy “gap” $\Delta(p,\omega)$ as a function of the momentum $p$ and the frequency $\omega$ is a generalized order parameter of a superconductor and is determined by solving the Eliashberg equations.\(^{(1)}\) In experiment, the average value of the “gap” $\Delta(\omega) = \langle \Delta(p,\omega) \rangle_p$ is usually determined from the tunnel characteristics of poly-
crystals.\(^{(2)}\) It is known that $\Delta(\omega)$ has singularities as a function of the parameter $\omega$.\(^{(2)}\) It was proposed\(^{(3)}\) that the singularities of $\Delta(\omega)$ can be due only to an irregular addition to the density of the number of states of the phonons when the topology of the phonon equal-energy surfaces is altered. The different equal-energy surfaces of the phonons manifest themselves in the external parameter $\omega$. Naturally, the singularities of $\Delta(\omega)$ are a consequence of the singularities of $\Delta(p,\omega)$. The nature of the possible singularities of $\Delta(p,\omega)$ in $(p,\omega)$ space is investigated in this paper from a unified point of view.

It is known that the quantity $\Delta(p,\omega) = \phi(p,\omega)/Z(p,\omega)$ is obtained by solving the system of equations for the function $\phi(p,\omega)$ and the parameter $Z(p,\omega)$.\(^{(1,3,4)}\) The real and imaginary parts of the functions $\phi(p,\omega)$ and $Z(p,\omega)$ satisfy the Cauchy relation.\(^{(1,3,4)}\) Therefore to determine the singularities in $\Delta(p,\omega)$ it suffices to determine them for the imaginary parts of $\phi(p,\omega)$ and $Z(p,\omega)$. Since the structure of the singularities in $\phi(p,\omega)$ and $Z(p,\omega)$ is the same, we confine ourselves to a study of the singularities in $\phi(p,\omega)$.

Let for the sake of argument $\omega > 0$. Then, following\(^{(1,3,4)}\), we represent the function $\text{Im} \phi(p,\omega)$ in the form

\[ \text{Im} \phi(p,\omega) = \phi_0(p,\omega) + \frac{\phi_1(p,\omega)}{\omega} + \frac{\phi_2(p,\omega)}{\omega^2} + \cdots \]

where $\phi_0(p,\omega)$ is a continuous function of $\omega$ and $\phi_1(p,\omega)$, $\phi_2(p,\omega)$,... are functions that are continuous in $\omega$ for $\omega \rightarrow 0$. The functions $\phi_1(p,\omega)$, $\phi_2(p,\omega)$,... have singularities in $\omega$ for $\omega \rightarrow 0$.
\[
I \text{m } \phi(p, \omega) = - \int \frac{d \omega'}{2T} \: \frac{1}{2} \frac{d \mu'}{(2\pi)^3} \: \text{Im} \left\{ \frac{\phi(p', \omega')}{Q(p', \omega')} \right\} \sum \frac{|g^\lambda_{\nu, p}|^2 \delta(\omega - \omega')}{\lambda} \sum \frac{d \lambda^\lambda_{p', p}}{2} \: \delta(p - p') - \omega - \omega',
\]
\[
Q(p, \omega) = \omega^2 Z^2(p, \omega) - \phi^2(p, \omega) - \xi^2(p); \: \xi(p) = \epsilon(p) - \mu.
\]

Here \(\epsilon(p)\) is the renormalized electron energy, \(g\) is the electron-phonon interaction parameter, \(\mu\) is the chemical potential, \(T\) is the temperature, and \(\lambda\) is the number of the branch of the phonon spectrum. To investigate the singularities in \(\phi(p, \omega)\), it is convenient to change over from integration with respect to \(dp'\) to integration with respect to the variables \(\epsilon(p')\), \(\omega_\lambda(p - p')\), and the length \(l_\lambda^p\) of the contour of the line at which the surfaces \(\epsilon(p') = \epsilon\) and \(\omega_\lambda(p - p') = \omega\) intersect (hereafter the "\(l_\lambda^p\) line" for brevity), and forego the transition to the variables \(\epsilon(p')\) and an arbitrary orthogonal coordinate system on the \(\epsilon(p') = \epsilon\) surface, which is customarily employed in superconductivity theory.\(^{11,13,4}\) In the general case, there may be several such \(l_\lambda^p\) lines, owing to the multiple-valuedness and periodicity of the functions \(\epsilon(p')\) and \(\omega_\lambda(p - p')\). We note that the length of the \(l_\lambda^p\) line depends on the external parameters \(p\) and \(\omega\). Carrying out the integration in (1), we obtain an expression for \(I \text{m } \phi(p, \omega)\) in the form

\[
I \text{m } \phi(p, \omega) = \frac{\omega}{2T} \: \sum \frac{d l^\lambda_{p, p}}{\lambda} \: \frac{|g^\lambda_{\nu, p}|^2}{|S^\lambda_{p - p}(\omega - \omega')||v_p(\mu)||\sin \theta_p|}
\]
\[
\times \text{ Re} \left\{ \frac{\Delta(p', \omega')}{\sqrt{\omega^2 - \Delta^2(p', \omega')}} \right\}
\]

Here \(\theta_p\) is the angle between the vectors \(v_p(\epsilon) = v_p(\epsilon(p)|_{\epsilon(p) = \epsilon}\) and \(S^\lambda_q(\omega) = \nabla q \omega_\lambda(q)|_{\omega_\lambda(q) = \omega}\) and depends on \(p\) and \(\omega\).

It follows from (2) that if the external parameters \(p\) and \(\omega\) are varied, then \(\phi(p, \omega)\) can have at \(p = p^*\) and \(\omega = \omega^*\) the following singularities: (1) at the points \(p' = p_0\), at which \(\sin \theta_{p*} = 0\), (2) at the points \(p' = p^* - q_k\), at which \(S^q_{p*}(\omega^*) = 0\). The geometrical interpretation of these cases is the following. In the first case the surfaces \(\epsilon(p_0) = \mu\) and \(\omega_\lambda(p^* - p_0) = \omega^*\) are tangent at the point \(p_0\). This is the analog of the Kohn singularity.\(^{15-7}\) The peculiarity of the Kohn singularities lies in this case in the fact that, besides satisfaction of the usual condition \(q = 2p^*_{\text{extr}}\), it is necessary that the vectors \(v_p(\mu)\) and \(S^q(\omega)\) be parallel or antiparallel (\(2p^*_{\text{extr}}\) is the extremal diameter of the Fermi surface). These conditions are satisfied, for example, in the case \(q_{\text{extr}} = 2p^*_{\text{extr}}\) (\(2q_{\text{extr}}\) is the extremal diameter of the surface \(\omega_\lambda(q) = \omega^*\)). In the second case, the surface \(\epsilon(p^*) = \mu\) crosses the surface \(\omega_\lambda(q) = \omega^*\) near directions of \(q_k\), where the topology of the phonon equal-energy surfaces changes. The topological singularities of the phonon spectrum manifest themselves if the surfaces \(\epsilon(p^*) = \mu\) and \(\epsilon(p^* - q_k) = \mu\) have either an intersection line or a tangency line.

580  JETP Lett., Vol. 27, No. 11, 5 June 1978  V.I. Makarov  580
In the cases considered, the $l_{p}^{\lambda}$ line changes its topology in an infinitesimally small vicinity of the points $p'=p_0$ and $p'=p^{*}-q_k$. The most characteristic topological changes are those in which the $l_{p}^{\lambda}$ either degenerates to a point (elliptic point) or has a self-intersection point (hyperbolic point).

We note that the singularities of the integral with respect to $dl_{p}^{\lambda}$ are similar to the singularities of the absorption coefficient of ultrasound in metals.\(^7\,8\)

The calculations show that at $p=p^{*}$ and $\omega=\omega^{*}$ the integral with respect to $dl_{p}^{\lambda}$ has a singular part for which the analytic expressions depends essentially on the character of the topological singularities of the $l_{p}^{\lambda}$ line. We note that integration with respect to $\omega$ does not lower the singularities of the integral with respect to $dl_{p}^{\lambda}$, because of the root singularity of the integrand $[\omega^{2}-A^{2}(p,\omega)]^{1/2}$, which is proportional to the density of the number of states of the quasiparticles in the superconductor.

To establish the form of the singularity in $\text{Im}\phi(p,\omega)$, we calculate its derivative

$$
\frac{\partial \text{Im} \phi(p^{*},\omega)}{\partial \omega} \sim \begin{cases} 
\pm \left[ (\omega - \omega^{*})^{2} - \Delta^{*2} \right]^{-1/2} \theta \left[ (\omega^{*} + \Delta^{*}) - \omega \right] \\
-\left[ \Delta^{*2} - (\omega - \omega^{*})^{2} \right]^{-1/2} \theta \left[ \omega - (\omega^{*} + \Delta^{*}) \right]
\end{cases},
$$

where $\Delta^{*}=\Delta(p',\Delta(p'))$ is obtained by solving the equation $\omega^{2}-\Delta^{2}(p',\omega)=0$, and the momentum $p'$ takes on a value either $p_0$ or $p^{*}-q_k$. For the sake of clarity we have presented the calculation results for two cases of topological singularities of the $l_{p}^{\lambda}$ line—the upper line of (3)—elliptic point ("+" corresponds to appearance of the $l_{p}^{\lambda}$ line with increasing $\omega$, and "−" to its vanishing), and the lower line corresponds to the hyperbolic point. It follows from (3) that $\partial \text{Im} \phi(p,\omega)/\partial \omega$ has a root singularity at $\omega=\omega^{*}+\Delta^{*}$. Using the Cauchy relation, we can show that the singular part of $\partial \text{Re} \phi(p,\omega)/\partial \omega$ also has a root singularity.

Thus, according to the foregoing analysis, in the general case the quantity $\Delta(p,\omega)$ can have in $(p,\omega)$ space singularities connected with the change of the topology of the line of intersection of the Fermi surface with the phonon equal-energy surfaces. These singularities\(^{11}\) in $\Delta(p,\omega)$ lead to a larger class of singularities in $\Delta(\omega)$ than hitherto assumed. We note that the singularities of $\Delta(\omega)$, which are due to the change of the topology of the $l_{p}^{\lambda}$ line near $q_k$, are of the same form as in\(^{11}\) only when the surfaces $e(p)=\mu$ and $e(p-q_k)=\mu$ have an intersection line.

If we consider as an external parameter, besides $p$ and $\omega$, also the pressure $P$, then in phase transitions of order $2^{1/3}$\(^{9}\) the quantity $\Delta(p,\omega)$ acquires in $(p,\omega)$ space singularities due to the change of the topology of the $l_{p}^{\lambda}$ line near the directions $p'=p_k$, where the topology of the Fermi surface changes. These singularities will be observed at $P\approx P^{*}$ for all values of $p$ and $\omega$ satisfying the conditions $\omega \lambda(p-p_k)=\omega$ and $e(p)=\mu$. The value $P^{*}$ corresponds to the pressure when $|\mu-\epsilon_k|=\omega<mS^{2}/2$, where $m$ is the electron mass, $S$ is the speed of sound, and $\epsilon_k=e(p_k)$ is the critical energy.
The considered singularities in the energy “gap” $\Delta(p,\omega)$ can apparently be observed in experiments on tunnel effects in single-crystal superconductors. Observation of these singularities would uncover new possibilities in the investigation of the singularities of both the phonon and the electron spectra of superconductors.

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**Curie law in diamagnetic susceptibility**

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The magnetic susceptibility of systems that are unstable to electron-hole pairing is investigated in the self-consistent-field approximation. It is shown that the response of the system is diamagnetic, and that its temperature dependence obeys the Curie law in the case of an imaginary order parameter.

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1. The purpose of this paper is to determine the directions in which searches can be made for systems with superdiamagnetic properties. It will be shown that for substances whose temperature of the transition to the current state $T_{Im}$ is higher than the temperature of the transition to the ferroelectric state $T_{Re}$, the diamagnetic susceptibility $\chi'$ obeys the Curie-Weiss law. For substances having $T_{Re}$ larger than $T_{Im}$, the diamagnetic susceptibility has a peak near $T_{Re}$. This peak should be observable if $T_{Re}$ and $T_{Im}$ have close values, and this is possible for ferroelectrics in which the ion contribution to the polarizability is not too large compared with the electron contribu-