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Fundamental length scale of quantum spacetime foam

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It is argued that the fundamental length scale for the quantum dynamics of spacetime need not be equal to the Planck length. Possibly, this new length scale is related to a nonvanishing cosmological constant or vacuum energy density.

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I. Introduction. It is generally assumed that the structure of a quantum spacetime foam [1 – 4], if physically relevant, is given by a *single* fundamental length scale, the Planck length $l_{\text{Planck}} \equiv \sqrt{\hbar G/c^3}$. In this Letter, we point out that a pure quantum spacetime foam ("pure" meaning independent of the direct presence of matter or nongravitational fields) could have a fundamental length scale l different from $l_{\text{Planck}} \propto \sqrt{G}$. Heuristically, the quantum spacetime foam would arise from gravitational self-interactions which need not involve Newton's constant G describing the gravitational coupling of matter (similar to the case of a gas of instantons in Yang–Mills theory [5], where the existence of gauge-field instantons does not depend on the coupling to additional fermion fields).

Henceforth, we adopt the conventions of Ref. [6], set the velocity c of light *in vacuo* to unity, and consider spacetime manifolds without boundaries (so that boundary terms in the action need not be considered explicitly [4]). The "matter" Lagrangian density is denoted by $\mathcal{L}_{\text{M}}(x)$ and we follow Einstein [7] by introducing a nonzero (positive) cosmological constant λ with the dimensions of an inverse length square. The standard classical action of general relativity (GR) is then given by [6]:

$$S^{\text{standard}}_{\text{gravitation}} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{|g(x)|}\,\big(R(x)+2\lambda\big) + \int d^4x\,\sqrt{|g(x)|}\,\mathcal{L}_{\text{M}}(x), \tag{1}$$

in terms of the Ricci scalar $R(x) \equiv g_{\mu\nu}(x)\,R^{\mu\nu}(x)$ and the scalar density $g(x) \equiv \det g_{\mu\nu}(x)$. Varying the metric $g_{\mu\nu}(x)$, the stationary-action principle gives the Einstein field equations [6, 7]:

$$R^{\mu\nu}(x) - \frac{1}{2}\,g^{\mu\nu}(x)\,R(x) - \lambda\,g^{\mu\nu}(x) = -8\pi G\,T^{\mu\nu}(x), \tag{2}$$

where the energy-momentum tensor $T^{\mu\nu}(x)$ of the matter is defined by the following functional derivative:

$$T^{\mu\nu}(x) \equiv \frac{2}{\sqrt{|g(x)|}}\;\frac{\delta\left(\int d^4y\,\sqrt{|g(y)|}\,\mathcal{L}_{\text{M}}(y)\right)}{\delta g_{\mu\nu}(x)}. \tag{3}$$

The experimental tests of GR [6, 8] only refer to solutions of the classical field equations (2). The reason is that classical gravity is already extraordinarily weak, so that even smaller quantum corrections are totally out of reach experimentally. Still, there are certain theoretical investigations which go beyond the classical theory but they essentially correspond to a semiclassical version of (2), with $T^{\mu\nu}$ on the right-hand side replaced by a (renormalized) vacuum expectation value of the appropriate expression in terms of quantum fields defined over a classical spacetime background [9]. Thus, as far as gravity is concerned, we can only be sure of the classical field equations (2).

The standard form of the action has the following defining property. Setting the cosmological constant to zero, $\lambda = 0$, the action (1) for Minkowski spacetime $g_{\mu\nu}(x) = \text{diag}(-1, 1, 1, 1)$ reduces precisely to the special-relativity action of the matter fields (in manifest agreement with the Equivalence Principle [10]). But, in this Letter, we are *only* interested in gravitational effects, taking the classical matter fields for granted. Moreover, we intend to stay completely agnostic as regards the quantum effects of spacetime and take the cosmological constant λ, viewed as a geometrical effect, to be strictly nonzero.

II. Generalized action. Even though the theory of "quantum gravity" does not exist, we can try to make some general remarks starting from the rescaled action which enters the quantum world of probability amplitudes via the Feynman [11] phase factor $\exp(\mathrm{i}\mathcal{I})$, with $\mathcal{I} = S/\hbar$ expressed in terms of the classical action S and the reduced Planck constant $\hbar \equiv h/2\pi$. This phase factor would, for example, appear in a properly defined

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path integral; see Ref. [4] for a discussion in the Euclidean framework.

The fact remains, however, that our understanding of the merging of quantum mechanics and gravitation is far from complete; see Ref. [12] for a clear account of at least one open problem. It may, therefore, be sensible to consider a *generalized* dimensionless action as possibly being relevant to a future quantum-gravity theory:

$$\mathcal{I}^{\,\text{generalized}}_{\,\text{gravitation}} = -\frac{1}{16\pi\, l^2}\int d^4x\,\sqrt{|g(x)|}\,\big(R(x)+2\lambda\big) + \\ + \frac{G/c^3}{l^2}\int d^4x\,\sqrt{|g(x)|}\;\mathcal{L}_{\text{M}}(x), \qquad (4)$$

where l is a new fundamental length scale and c has been temporarily restored. To emphasize once more, the generalized action (4) is only to be used for a rough description of possible quantum effects of spacetime, not those of the matter fields. Specifically, the matter fields which enter $\mathcal{L}_{\text{M}}(x)$ in (4) are considered to act as fixed classical sources and cannot be rescaled. In fact, $\hbar$ does not appear at all in (4), as will be discussed further below.

Setting l equal to the Planck length obtained from Newton's gravitational constant G, the light velocity c, and Planck's quantum of action $h \equiv 2\pi\,\hbar$,

$$l_{\text{Planck}} \equiv \sqrt{\hbar\, G/c^3} \approx 1.6\cdot 10^{-35}\,\text{m}\;, \qquad (5)$$

the standard form (1) of the action is reproduced (taking again $c = 1$):

$$\mathcal{I}^{\text{generalized}}_{\text{gravitation}}\Big|_{l=l_{\text{Planck}}} = S^{\text{standard}}_{\text{gravitation}}/\hbar. \qquad (6)$$

Yet, the generalized action (4) with an independent length scale l looks more natural, as Newton's constant G multiplies the classical source term. The overall factor l^{-2} in (4) is, of course, irrelevant for obtaining the classical field equations (2). But, as stressed by Feynman [11], quantum physics is governed by the phase factor $\exp(\mathrm{i}\,\mathcal{I})$ and, for a genuine quantum spacetime, the overall factor of l^{-2} in (4) would be physically relevant.

The expression (4) for the gravitational quantum phase also suggests that, as far as spacetime is concerned, the role of Planck's constant $\hbar$ would be replaced by the squared length l^2, which might loosely be called the "quantum of area." Planck's constant $\hbar$ would continue to play a role in the description of the matter quantum fields. But, with $\hbar$ and l^2 *logically independent*, it is possible to consider the "limit" $\hbar \to 0$ (matter behaving classically) while keeping l^2 fixed (spacetime behaving nonclassically), which would correspond to the applicability domain of the generalized action (4). At the level of *Gedankenexperiments*, the quantum spacetime resulting from (4) could then be studied with classical measuring rods and standard clocks.

The table below summarizes the (overcomplete) set of fundamental dimensionful constants at our disposal. The scope of this Letter has deliberately been restricted to a qualitative discussion of the second and third columns of the table, leaving the unified and rigorous treatment of *all* columns to a future theory. For example, the Hawking temperature of a Schwarzschild black hole with mass M is given by $T_H = \hbar\, c^3/8\pi k G M$ and lies outside the scope of the present Letter ($\hbar = 0$). But a future quantum-gravity theory (possibly with fundamental constants $\hbar$, c, G, and l^2) would certainly have to explain black-hole thermodynamics [13].

Fundamental constants of nature, including the hypothetical "quantum of area" l^2

Quantum matter	Classical relativity	Quantum spacetime
$\hbar$	$c\,,\,G$	l^2

Referring to the table, two parenthetical remarks can be made, one on the "quantum of area" entering the third column and another on a possible type of theory for all of the columns of the table. First, observe that loop-quantum-gravity calculations [14, 15], with l_{Planck} replaced by l, give for a subset of the eigenvalue spectrum of the area operator the values $8\pi\gamma\, l^2 \sum_{i=1}^{n}\sqrt{j_i\,(j_i+1)}$, with positive half-integers j_i, positive integers n, and a real parameter $\gamma > 0$. Very briefly, j_i labels the irreducible $SU(2)$ representation of spin-network link i intersecting the test surface $\mathcal{S}$ and the sum over i builds up the area of $\mathcal{S}$, with the (somewhat mysterious) Barbero–Immirzi parameter γ entering the definition of the quantum theory. The smallest eigenvalue in this subset is apparently given by $4\pi\sqrt{3}\,\gamma\, l^2$, which would then correspond to the precise value of the quantum of area. The fundamental role of a quantum of *area* would be inline with the suggested "holographic principle" [16, 17].

Second, observe that, with an extra fundamental length l available in the table, it would be possible to eliminate, for example, Newton's gravitational constant G by writing $G = f\, c^3\, l^2/\hbar$ with a numerical factor f. This trivial exercise may hint at a new type of induced-gravity theory [18, 19], with "classical gravitation" emerging from the *combined* quantum effects of matter and spacetime, so that the number f may be calculable. In fact, the magnitude of the Newtonian gravitational acceleration towards a macroscopic point mass M at a macroscopic distance D can be written in the following suggestive form: $GM/D^2 = f\, c\,(Mc^2/\hbar)\,(l^2/D^2)$, with all microscopic quantities indicated by lower-case symbols. Remarkably, the possible appearance of a new parameter in quantum-gravity theory has also been suggested by a hydrodynamics analogy [20].

Returning to the earlier discussion of l and l_{Planck}, the only conclusion, for the moment, is that it may be important to recognize the length scale l as a free parameter of the quantum theory of spacetime, which is not directly related to l_{Planck}. Putting aside epistemological issues, there are then two cases to discuss: l larger than or equal to l_{Planck} and l smaller than l_{Planck}. The first case ($l \geq l_{\text{Planck}}$) appears to be a serious physical possibility. Presupposing the existence of a classical spacetime manifold for reference, l_{Planck} would correspond to the minimal length which could, in principle, be measured by a massive particle obeying the Heisenberg position–momentum uncertainty relations; cf. Refs. [21, 22]. But, as the probed distances are made shorter and shorter (starting at the atomic scale, for example), it could be that the classical spacetime picture breaks down *well before* the length l_{Planck} is reached, the breakdown occurring at distances of order l (assuming, for the moment, that $l \gg l_{\text{Planck}}$).

The second case ($l < l_{\text{Planck}}$) looks, at first, rather academic as long as all material probes of spacetime interact with a coupling strength G. However, it could be (possibly in an extended version of the theory) that a sub-Planckian spacetime structure determines certain effective parameters for the physics over distances of the order of l_{Planck} or larger. An analogy would be atomic physics, which determines the electric permittivity ϵ and magnetic permeability μ controlling the propagation of electromagnetic waves with wavelengths very much larger than the atomic length scale.

In either case, the main outstanding problem is to understand how the *quantum* spacetime foam gives rise to an effective *classical* spacetime manifold over large enough distances, perhaps requiring an objective state-reduction mechanism (cf. Refs. [12, 23]). But, given our daily experience, we can be sure that there must be some kind of "crystallization" of classical spacetime, with or without the occasional "defect" in the resulting structure.

Incidentally, a nontrivial classical spacetime-foam remnant can be expected to lead to an effective violation of the Weak Equivalence Principle (WEP) [8] on the following grounds. Lorentz invariance of a massless spin–2 particle (graviton) essentially implies WEP [10, 24]. But soft gravitons propagating over a classical spacetime-foam remnant are believed to have Lorentz violation (for example, from a modified dispersion relation) and, therefore, WEP is no longer guaranteed to hold. Moreover, particles of different spin can be expected to propagate differently over a classical spacetime with nontrivial small-scale structure, because of nontrivial "boundary conditions" on the particle fields. Precisely these type of propagation effects can lead to interesting astrophysical bounds, as will be discussed further at the end of the next section.

III. Conjecture. The crucial question, now, is what the value of the fundamental length scale l really is, compared to l_{Planck}? At this moment, the answer is entirely open and $l = l_{\text{Planck}}$ is certainly not excluded. Let us, here, present a conjecture suggested by cosmology. Henceforth, we set $c = \hbar = k = 1$, with k Boltzmann's constant.

Recent astronomical observations seem to indicate a nonzero value of the cosmological constant, $\lambda_0 \equiv \equiv L_0^{-2} > 0$, with a length scale L_0 corresponding to the size of the visible universe, $L_0 \approx 10^{10}\,\text{ly} \approx 10^{26}\,\text{m}$; see Refs. [25, 26] and references therein. In addition, it is possible that the universe at an early stage had a significant vacuum energy density, $\rho_{\text{vac}} \equiv E_{\text{vac}}^4$; see, e.g., Ref. [27] for background on so-called "inflation" models. Setting λ in (4) equal to λ_0 and $\mathcal{L}_{\text{M}}(x)$ to $-\rho_{\text{vac}}$ (that is, neglecting the kinetic terms of the fields in first approximation), one has

$$\mathcal{I}_{\text{gravitation}}^{\text{generalized}} \sim -\frac{1}{16\pi\, l^2} \int d^4x\, \sqrt{|g(x)|} \times \times \big(R(x) + 2\lambda_0 + 2\Lambda\big), \tag{7}$$

with the effective cosmological constant

$$\Lambda \equiv 8\pi\, l_{\text{Planck}}^2\, \rho_{\text{vac}} = 8\pi\, E_{\text{vac}}^4 / E_{\text{Planck}}^2\,, \tag{8}$$

in term of the energy scales E_{vac} and $E_{\text{Planck}} \equiv \equiv 1/l_{\text{Planck}} \approx 1.2 \cdot 10^{19}\,\text{GeV}$.

For the case $\Lambda \gg \lambda_0$ (possibly relevant to the early universe), it may be natural to have the lengths l and $\sqrt{3\pi/\Lambda}$ of the same order, where the numerical factor $\sqrt{3\pi}$ has been introduced to streamline the discussion. The de-Sitter instanton [4, 5], for example, gives for the right-hand side of (7) an absolute value of $3\pi/(l^2\,\Lambda)\,[1 + O(\lambda_0/\Lambda)]$, which would then be of order unity. As mentioned in the Introduction, the quantum spacetime foam would somehow resemble a gas of these gravitational instantons.

For the case $0 \leq \Lambda \leq \lambda_0$ (possibly relevant to the final state of the universe), the same argument suggest that the fundamental length scale l would be of the order of $\sqrt{3\pi/\lambda_0} \approx 3 \cdot 10^{26}\,\text{m}$. Our galaxy would then be buried *deep inside* a quantum-spacetime-foam remnant with typical length scale $l \sim L_0$, provided the effective classical spacetime manifold can be smoothed over distances less than l. The resulting picture of the universe would, in a way, resemble that of Linde's "chaotic inflation" [27, 28] but have a different origin.

Returning to the case $\Lambda \gg \lambda_0$, the conjecture is, therefore, that the fundamental length scale l can be significantly larger than the Planck length and have the following order of magnitude:

$$l \stackrel{?}{\sim} E_{\text{Planck}}/E_{\text{vac}}^2 \approx$$
$$\approx 1.6 \cdot 10^{-29}\,\text{m} \left(\frac{E_{\text{Planck}}}{10^{19}\,\text{GeV}}\right) \left(\frac{10^{16}\,\text{GeV}}{E_{\text{vac}}}\right)^2, \qquad (9)$$

where the numerical value for E_{vac} has been identified with the "grand-unification" scale suggested by elementary particle physics [29]. (The energy scale $E_{\text{Planck}} \equiv \equiv 1/l_{\text{Planck}} \gg 1/l$ would continue to play a role in, for example, the ultra-high-energy scattering of massive neutral particles by graviton exchange.) In terms of the fundamental constants G and l from the generalized action (4), the vacuum energy density would be given by (temporarily reinstating c)

$$\rho_{\text{vac}} \stackrel{?}{\sim} c^4/G\,l^2, \qquad (10)$$

which would suggest a gravitational/spacetime origin for this vacuum energy density ($\hbar$ not appearing directly; cf. Table).

Remark that the detailed quantum structure of spacetime may very well involve *two* length scales, here taken to be l_{Planck} and l. Indeed, if (9) holds true with $l_{\text{Planck}}/l \sim 10^{-6}$, it is perhaps possible to have sufficiently rare spacetime-foam defects left-over from the crystallization process mentioned in the penultimate paragraph of Sec. II. With average spacetime defect size $\widetilde{b}$ set by l_{Planck} and average separation $\widetilde{l}$ set by l, these time-dependent defects would be close to saturating the current astrophysical bounds from ultra-high-energy cosmic rays, the so-called excluded-volume factor in the modified photon dispersion relation being $(\widetilde{b}/\widetilde{l})^4 \sim 10^{-24}$; see Ref. [30] for details. (The present Letter originated, in fact, from an attempt to understand these astrophysical bounds.)

Equally important, with $l \gg l_{\text{Planck}}$, the dynamics of the early universe for temperatures $T \gtrsim 1/l \stackrel{?}{\sim} 10^{13}$ GeV would have to be reconsidered as a classical spacetime description might be no longer relevant. For sufficiently low temperatures ($T \ll 1/l$), where a classical spacetime description should be valid, the expansion of the universe would still be given by the standard Einstein equations (2) and corresponding Friedmann equations [6], as the generalized quantum action (4) has been designed not to modify the basic classical field equations.

In this section, we have presented only one particular conjecture for the fundamental length scale l. But the idea is more general, namely that entirely new physics may be responsible for the small-scale structure of spacetime.

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