CONFORMAL SYMMETRY OF CRITICAL FLUCTUATIONS

A.M. Polyakov
L.D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
Submitted 26 October 1970
ZhETF Pis. Red. 12, No. 11, 538 - 541 (5 December 1970)

In [1, 2] it was suggested that the fluctuations are scale-invariant at a phase transition point. This hypothesis was confirmed in [3, 4] by field-theoretical methods.

We show in the present paper that the correlation functions at the transition point are invariant against transformation of a conformal group that includes a change of scale as a particular case. This circumstance makes it possible to calculate in explicit form any three-point correlators and greatly limit the possible form of multipoint correlators.

The conformal group[1)] consists of the following transformations:

$$x' = \lambda x, \quad \frac{x'}{x'^2} = \frac{x}{x^2} + \vec{\alpha}, \quad x'_i = a_{ik}x_k + b_i. \tag{1}$$

The last equation describes usual rotations and displacements, and the first describes scale transformations. The second equation is called the special conformal transformation. For obvious reasons, we shall investigate just this transformation, and will find it convenient to assume that $\vec{\alpha}$ is infinitesimally small. Then the transformation is written in the form

$$x' - x \equiv \delta x = x^2\vec{\alpha} - 2(\vec{\alpha}x)\,x. \tag{2}$$

From (2) we easily obtain the formula

$$\delta \ln r_{AB} = -\vec{\alpha}(x_A + x_B), \tag{3}$$

where

$$r_{AB} = \sqrt{(x_A - x_B)^2}.$$

The meaning of the transformation (2) is that at each point of space there occurs an infinitesimally small isotropic stretching by a factor $\lambda(x)$, where

$$\lambda(x) = 1 - 2\vec{\alpha}x + O(\alpha^2). \tag{4}$$

Formula (4) follows from (3) when $x_A \to x_B$.

From the results of [1 - 4] it follows that in the case of homogeneous small stretching by a factor $\lambda = 1 + \varepsilon$, any fluctuating quantity ϕ should change in the following manner:

$$\begin{cases} \phi(x) \to \phi'(x') = \lambda^{-\Delta_\phi}\phi(x) \approx \phi(x) - \Delta_\phi \epsilon\,\phi(x), \\ x' = x + \epsilon x \end{cases} \tag{5}$$

[1)] For a review of the properties of this group in Lagrangian field theory see [5].

where the constant Δ_ϕ is the unknown critical index of the field ϕ. The obvious generalization of (5) for conformal transformations is

$$\delta\phi(x) \equiv \phi'(x') - \phi(x) = 2\Delta_\phi(\vec{a}x)\phi(x). \tag{6}$$

It will be shown below that the field-theory equations are invariant against the equal-time transformations (2) and (6), and therefore the correlation functions are not altered by these transformations.

Before we present the proof, let us investigate the consequences of the statements made above.

We consider a three-point correlator of arbitrary fields a, b, c with dimensionalities Δ_a, Δ_b, and Δ_c

$$G_{III}(r_{12}, r_{13}, r_{23}) = \langle a(r_1) b(r_2) c(r_3) \rangle. \tag{7}$$

The conformal transformation shifts the points $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ and the relative distances r_{ik} change in accordance with (3). The corresponding change of G is

$$\delta G_{III} = -\sum_{i,k} \frac{\partial G_{III}}{\partial \ln r_{ik}} \vec{a}(r_i + r_k). \tag{8}$$

This quantity should equal the change of G as a result of the transformation (5)

$$\delta G_{III} = 2\sum \Delta_i \vec{a}\, r_i\, G_{III}. \tag{9}$$

Equating the coefficients in front of $\alpha\vec{r}_i$ in (9) and (8) we obtain a simple system of equations for G_{III}, the unique solution of which is

$$G_{III} = \mathrm{const}\; r_{12}^{(\Delta_c - \Delta_a - \Delta_b)}\, r_{13}^{(\Delta_b - \Delta_c - \Delta_a)}\, r_{23}^{(\Delta_a - \Delta_b - \Delta_c)}. \tag{10}$$

Formula (10) is confirmed in the flat Ising model, where an expression for the correlator $\langle\varepsilon\sigma\sigma\rangle$ is known, where ε is the energy density and σ is the magnetic moment [6, 7]. For a four-point function, analogous arguments yield the result

$$G_{IV} = \left(r_{12}^{\Delta_b + \Delta_d}\, r_{24}^{\Delta_a + \Delta_c}\right)\left(r_{12}^{-\Delta_a - \Delta_b}\, r_{23}^{-\Delta_b - \Delta_c}\right)\left(r_{34}^{-\Delta_c - \Delta_d}\, r_{41}^{-\Delta_d - \Delta_a}\right) \times \times F\left(\frac{r_{13}r_{24}}{r_{12}r_{34}}, \frac{r_{14}r_{23}}{r_{12}r_{34}}\right), \tag{11}$$

where F is an arbitrary function. In the case of N-point correlators, the conformal invariance leaves a function of [N(N - 3)]/2 variables invariant.

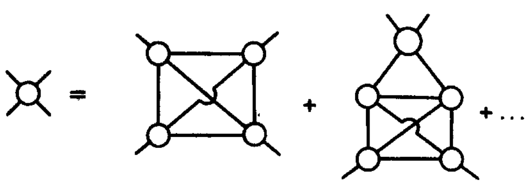

Fig. 1

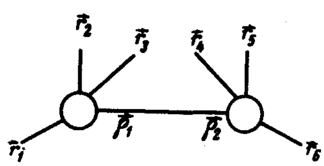

Fig. 2

The field-theory equations describing the phase transitions express the exact amplitudes in terms of one another [3, 4]. One such equation is shown in Fig. 1, where the lines correspond to the correlators of the magnetic moment. If the series in Fig. 1 converges then, as proved in [3], the equations admit of a group of scale transformations and lead to the physical picture of [1, 2]. We shall show that under the same assumptions the equations are conformally invariant, i.e., amplitudes of the type (10) and (11) duplicate themselves when substituted in the diagrams. For the proof, it suffices to verify that any element of the diagram containing exact amplitudes, for example the one shown in Fig. 2, is multiplied by the quantity $2\Sigma\Delta_i(\vec{\alpha}\cdot\vec{r}_i)$ when the coordinates of the end points of $\vec{r}_i$ are changed in accordance with (2).

In the coordinate representation, the contribution of Fig. 2 can be written in the form

$$\int d^3\vec{\rho}_1 d^3\rho_2 G_{IV}(r_1 r_2 r_3 \vec{\rho}_1) H(\vec{\rho}_1 - \vec{\rho}_2) G_{IV}(\vec{\rho}_2 r_4 r_5 r_6) \quad (12)$$

where

$$\left[\int H(\vec{\rho}-\vec{\rho}_1) G_{II}(\vec{\rho}_1 - \vec{\rho}\,') d^3\rho_1 = \delta(\vec{\rho}-\vec{\rho}\,')\right],$$

G_{IV} and G_{II} are correlators of four and two magnetic moments.

Let us displace the points $\vec{r}_i$ in accordance with (2) and simultaneously make the following change of variables in (12):

$$\vec{\rho}_i\,' = \vec{\rho}_i + \alpha\vec{\rho}_i^{\,2} - 2(\vec{\alpha}\vec{\rho}_i)\vec{\rho}_i \,. \quad (13)$$

The integral (12) changes because

$$G_{IV}' = G_{IV}(1 + 2\Delta\Sigma\vec{\alpha} r_i + 2\Delta\vec{\alpha}\vec{\rho}),$$

$$H' = H\left[1 + 2(3-\Delta)\vec{\alpha}(\vec{\rho}_1 + \vec{\rho}_2)\right] \,, \quad (14)$$

$$d^3\rho_1' d^3\rho_2' = d^3\rho_1 d^3\rho_2 \left[1 - 6\vec{\alpha}(\vec{\rho}_1 + \vec{\rho}_2)\right] \,.$$

Substituting (14) in (12), we see that the coefficients in front of $\vec{\alpha}\cdot\vec{\rho}_1$ and $\vec{\alpha}\cdot\vec{\rho}_2$ vanish and the entire integral is multiplied by $2\Delta\Sigma\vec{\alpha}\cdot\vec{r}_i$, q.e.d.

By building up more complicated diagrams out of the simple ones and repeating the preceding arguments, we can verify that each term of the series in Fig. 1 has conformal properties.

The author is grateful to A.I. Larkin, A.A. Migdal, and V.L. Pokrovskii for a discussion and to H. Kastrup (West Germany) for explaining the mathematics of the conformal group.

[1] A.Z. Patashinskii and V.L. Pokrovskii, Zh. Eksp. Teor. Fiz. 50, 439 (1966) [Sov. Phys.-JETP 23, 292 (1966)].
[2] L.P. Kadanoff, Physica 2, 263 (1966).
[3] A.M. Polyakov, Zh. Eksp. Teor. Fiz. 55, 1026 (1968) [Sov. Phys.-JETP 28, 533 (1969)].
[4] A.A. Migdal, ibid. 55, 1964 (1968) [28, 1036 (1969)].
[5] G. Mack and A. Salam, Ann. Phys. 53, 174 (1969).
[6] L.P. Kadanoff, Preprint (1969).