# Particle spectrum in quantum field theory

A. M. Polyakov

*L. D. Landau Institute of Theoretical Physics*
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We wish to call attention in this article to a hitherto uninvestigated part of the spectrum of ordinary Hamiltonians used in quantum field theory. As the first example we consider the model of a self-interacting scalar field in two-dimensional space-time, with Hamiltonian

$$H = \frac{1}{2}\int_{-\infty}^{+\infty} dx \left\{ \pi^2 + \left(\frac{d\phi}{dx}\right)^2 - \mu^2\phi^2 + {}^{\lambda}\!/_{2}\,\phi^4 \right\}$$
$$\pi(x) = -i\,{}^{\delta}\!/_{\delta\phi(x)} \quad . \tag{1}$$

In this model, the vacuum is filled with the Bose condensate $\bar{\phi}^2 = \mu^2/\lambda$. Over the vacuum there is a single-particle state with mass $\sqrt{2}\mu$, which represents small oscillations of the constant condensate. However, constant $\bar{\phi}$ is not the only stable equilibrium state. There is another extremal of the potential energy in (1), determined from the equation

$$\phi_c'' + \mu^2\phi_c - \lambda\phi_c^3 = 0 \tag{2}$$

with boundary conditions $\phi_c^2(\pm\infty) = \mu^2/\lambda$, which mean that the vacuum is perturbed only within a finite volume. The solution of (2) is

$$\phi_c(x) = \frac{\mu}{\sqrt{\lambda}}\,\mathrm{th}\,\frac{\mu x}{\sqrt{2}} \,. \tag{3}$$

To calculate the spectrum of the vibrational energy levels near the considered equilibrium point, we write

$$\phi(x) = \phi_c(x) + \phi(x)$$

and neglect the terms $\phi^3$ and $\phi^4$ in the Hamiltonian (the latter is valid if $\lambda \ll \mu^2$). Diagonalizing the obtained quadratic Hamiltonian, we obtain the mass spectrum

$$M_n = \frac{2\sqrt{2}\,\mu^3}{3\,\lambda} + \frac{\sqrt{3}+2}{2\sqrt{2}}\mu + \frac{\mu}{\sqrt{2}}\sqrt{n(4-n)}$$
$$n = 0,\ 1,\ 2\,. \tag{4}$$

In formula (4), the first term is the potential energy at the equilibrium point, the second is the zero-point oscillation energy, and the third is the excitation energy. Thus, in this model there are three types of particles with anomalously large masses. We call these objects "extremons."

In the generalized formalization, the results consists in the fact that each stationary regular solution of the classical equations of motion corresponds in quantum field theory with weak coupling to its own set of extremons, the masses of which can in principle be calculated. We shall show that extremons exist in three-dimensional models. We consider the theory of the Higgs isovector field $\phi_a(x)$, $a = 1, 2, 3$,

$$H = \int \left\{ \frac{1}{2}\pi_a^2 + \frac{1}{2}(\nabla\phi_a)^2 - \frac{1}{2}\mu^2\phi_a^2 + \frac{\lambda}{4}(\phi_a^2)^2 \right\} d^3x \,. \tag{5}$$

The equation for the extremal

$$\nabla^2\phi_a + \mu^2\phi_a - \lambda\sum_b \phi_b^2 \quad \phi_a = 0$$

has a solution

$$\phi_a = x_a u(r) r^{-1} \,,$$

where $u$ is subject to the equation

$$u'' + \frac{2}{r}u' + \left(\mu^2 - \frac{2}{r^2}\right)u - \lambda u^3 = 0$$
$$u(\infty) = \mu/\sqrt{\lambda}\,; \qquad u(r) = \mathrm{const}\ r\,.$$

We call this the "hedgehog" solution, inasmuch as the isovector at a given point of space is directed along the radius vector. The solitary hedgehog is not an extremon, since its energy diverges linearly at large distances, owing to the inhomogeneity of the distribution of the directions of the field $\phi_a$. There are two ways of overcoming this difficulty. The first is to connect to the hedgehog a Yang-Mills field, i.e., to make the substitution

$$\nabla_\mu\phi_a \to \nabla_\mu\phi_a + g\epsilon_{abc}A_\mu^b\phi_c$$

and to add the Yang-Mills Hamiltonian to (5). By virtue of the gauge symmetry of the second kind, the inhomogeneity of the directions then becomes physically unrealizable and makes no contribution to the energy, which therefore turns out to be finite. The solution of the classical equations is

$$\phi_a(x) = x_a u(r) r^{-1},$$
$$A_\mu^a(x) = \epsilon_{\mu ab}\, x_b \left(a(r) - \frac{1}{gr^2}\right), \tag{6}$$

where $u$ and $a$ satisfy the equations

$$\begin{cases} u'' + \dfrac{2}{r}u' + (\mu^2 - 2g^2a^2(r))\,u - \lambda u^3 = 0 \\ a'' + \dfrac{4}{r}a' - \dfrac{3}{r^2}a - g^2r^2a^3 - g^2u^2a = 0 \end{cases} \,.$$

The mass of the resultant extremon is of the order of

$$M \sim \frac{\mu^2}{\lambda}\, m_V^{-1} \sim m_V/g^2$$

(where $m_V = g^2u^2(\infty)$ is the mass of the vector bosons).

As is well known, the model under consideration has one massless vecton and two massive vectons. If the first of them is identified with the photon, then the hedgehog has by virtue of (6) a magnetic charge.[1)]

It is easy to construct hedgehogs in which all the components of the gauge field are massive and concentrated within the region $1/m_V$. To this end it suffices to consider an isotensor or isospinor Higgs field. In the former case the solution should be sought in the form

$$\phi_{ab} = r^{-2}(x_a x_b - \frac{1}{3}\delta_{ab} r^2)u(r)$$

and in the latter case it has a more complicated form and will be described elsewhere.

Another possibility for the construction of an extremon is formation of a hedgehog-antihedgehog pair. It is easy to show that the pair energy is $E = AR$, where $R$ is the distance between the pair components. To stabilize this state it is necessary to consider levels with zero angular momentum $L$

$$E_{eff} = AR + B\frac{L^2}{R^2}. \tag{7}$$

The mass spectrum is given by

$$M^2(L) = \text{const}\, L^{4/3}. \tag{8}$$

A rigorous justification of (7) calls for the solution of the equation for the extremal with allowance for the fixed angular momentum. We have derived such an equation, but are unable to solve it exactly. Therefore the validity of (8) depends on the hypothesis concerning the character of the hedgehog rotation. In particular, if it is assumed that the region of space contained between the hedgehogs takes part in the rotation, then we obtain in place of (8) the formula

$$M^2(L) = \text{const}\, L.$$

There is no doubt, however, that a formula of the type (8) for the energy should lead to growing Regge trajectories.

Thus, in the interpretation of the elementary-particle spectrum on the basis of field theory one should bear in mind the many new possibilities that are afforded by taking the extremon states into account.

Ideas very close to those described above were developed in[1-4] but it seems to me that both the analysis method and the results obtained above are new to some degree.

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[1)]This circumstance was pointed out to me by L.B. Okun'. After completing the work, I obtained a preprint by t'Hooft, which contains a similar result.

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