Derivation of exact spectra of the Schrödinger equation by means of supersymmetry

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A new type of hidden symmetry has been found. It can be used to find the complete spectra for a broad class of problems including all known exactly solvable problems of quantum mechanics through elementary calculations. How this symmetry can explain reflectionless potentials is shown.

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1. Supersymmetry$^1$ is attracting increasing interest among physicists, and its fields of application are far from being exhausted. In this letter we analyze the energy spectrum of a supersymmetry quantum mechanics, which is an important model for studying the structure of supersymmetry theories.$^2$ We derive the conditions under
which the problem of finding the complete spectrum has an exact and very simple
solution.

This result may be of interest in two ways. First, it suggests some new aspects of
exactly solvable models (the exact solution of the spectrum problem, which we are
discussing here, is related to a sort of hidden symmetry of the Hamiltonian). Second, it
simplifies the problem of finding the complete spectra for a broad class of one-dimen-
sional problems (or of problems which can be reduced to one-dimensional problems) in
ordinary quantum mechanics. In particular, this is true of all known exactly solvable
spectral problems. All such problems have a hidden symmetry, and this symmetry
makes the problem of calculating the spectra an elementary one; the answer can be
found almost immediately.

This approach also furnishes an explanation for the reflectionless nature of poten-
tials of the type \( U(x) = -n(n+1)/2\chi^2x \), which are important in the theory of soli-
tons: Such potentials are related by transformations of this symmetry with the poten-
tial \( U(x) = 0 \).

2. The Hamiltonian of our supersymmetry quantum mechanics is

\[
H = (p^2 + W^2(x) + \sigma_3 W'(x))/2
\]

(\( \sigma_i \) are the Pauli matrices) and acts on two-component wave functions. The supersym-
metry generators \( Q_1 = (\sigma_2 p + \sigma_2 W)/2; Q_2 = (\sigma_2 p - \sigma_1 W)/2 \) satisfy the algebra

\[
Q_i^2 = Q_i Q_7 = H/2; [Q_1, Q_2] = 0; [H, Q_i] = 0,
\]

making the spectrum of \( H \) non-
negative and the levels degenerate. The only level which may be nondegenerate is the
lowest, whose energy in this case is zero.

It follows that the two customary one-dimensional Hamiltonians \( H_{\pm} \),

\[
H_{\pm} = p^2/2 + (W^2(x) \pm W'(x))/2
\]

have identical spectra for an arbitrary function \( W(x) \). The only exceptional case may be
the lowest level of one of \( H_{\pm} \), in which case its energy is exactly zero. Below we use
these two properties of supersymmetry theories—the level degeneracy and the vanish-
ing ground-state energy—to find the exact spectrum.

3. We assume that \( H_{\pm} \) has a zero level [i.e., that \( \psi_0 = \exp(-J W(x')dx') \) is nor-
malizable].

How are the potentials \( U_{\pm} = (W^2 \pm W')/2 \) related? If they differ only in the
parameters which appear in them (including an additive constant), then the complete
spectrum of the Hamiltonians \( H_{\pm} \) and thus of the supersymmetry Hamiltonian \( H \) can
be found easily. To show this, we assume

\[
U_{\pm}(a, x) = U_{\pm}(a_1, x) + R(a_1),
\]

where \( a \) is the set of parameters, and \( a_1 = f(a) \).

We construct the series of Hamiltonians \( H_n, n = 0, 1, 2, ... \)

\[
H_n = p^2/2 + U_{\pm}(a_n, x) + \sum_{k=1}^{n} R(a_k),
\]

where \( a_n = f(a) \) (i.e., \( f \) is applied \( n \) times), and we compare the spectra \( H_n \) and \( H_{n+1} \).

Using (3), we find
Comparing expressions (4) and (5), and using the results of Section 2, we see that $H_n$ and $H_{n+1}$ have identical spectra except for the lowest level of $H_n$, whose energy is $\Sigma_{k=1}^n R(a_k)$, as follows from (4). In going from $H_n$ to $H_{n-1}$, etc., we get back our original Hamiltonian $H_0 = H_- = p^2/2 + U_-(a,x)$, whose lowest level is zero, and all of whose other levels coincide with the lowest levels of the Hamiltonians $H_n$. The complete spectrum $H_-$ is thus given by $\tilde{E}_n = \Sigma_{k=1}^n R(a_k)$. The spectrum of a Hamiltonian with a potential $U(a,x) = U_-(a,x) + C(a)$ is thus

$$E_n = \tilde{E}_n + C(a) = \Sigma_{k=1}^n R(a_k) + C(a).$$

Expression (6) is the basic result of this paper.

4. To demonstrate the use of this approach we consider the interesting example of the potential $U(a,x) = -a(a+1)/2c^{2}x$, which is known to be reflectionless for integer values of $a$. In this case we have $W(x) = a^2 x$. We have $U_-(a,x) = -a(a+1)/2c^{2}x + a^2/2$. Hence $a_1 = f(a) = a - 1, a_n = a - n, C(a) = -a^2/2$, and $R(a_k) = (a_k^2 - a_{k-1}^2)/2$, so that $\Sigma_{k=1}^n R(a_k) = (a^2 - a_n^2)/2$. The complete spectrum is then, according to (6),

$$E_n = -a_n^2/2 = -(a - n)^2/2.$$

The procedure for finding the spectra in all other cases (see the following section) is completely analogous and equally elementary.

In the example at hand, the potential $U(a,x)$ with integer $a$ is reduced by a sequence of transformations (3) to a potentials $U(x) \equiv 0$, since $a_n = a - n$. It is for this reason that potentials of this sort are reflectionless: The eigenfunctions of the Hamiltonians $H_n$ and $H_{n+1}$ are coupled by the action of the operators $Q_x \sim [d/dx \pm W(x)]$, which do not transform the functions $\exp(\pm ikx)$ into each other, and with $U(x) \equiv 0$ there is obviously no reflection.

5. If the potentials $U_\pm$ satisfy condition (3), i.e., if the function $W(a,x)$ satisfies the functional-differential equation

$$W^2(a,x) + W'(a,x) = W^2(a_1, x) - W'(a_1, x) + 2R(a_1),$$

then the spectra of the Hamiltonians $H_\pm$ and thus the spectrum of supersymmetry model (1) can be found by elementary calculations by the approach described here. We have found the following solutions of Eq. (7):

$$W = af_1 + b; \; W = af_2 + b/f_2; \; W = (a + b \sqrt{pf_3^2 + q})/f_3,$$

where the functions $f_1$, $f_2$, and $f_3$ satisfy the (separable-variable) differential equations

$$f'_1 = pf_1^2 + qf_1 + r, f'_2 = pf_2^2 + q, f'_3 = \sqrt{pf_3^2 + q}$$

with arbitrary constants $p, q, r$. The corresponding potentials incorporate all potentials for which exact spectra have been found so far (only eight such potentials were given in Ref. 3) and also several other potentials with the same qualitative behavior.
1) \( U(x) = \frac{a(a - 1) - b(b + 1)}{2 \sin^2 x} - \frac{b(2a + 1) \cos x}{2 \sin^2 x} \); 
2) \( U(x) = \frac{a(a + 1) + b^2}{2 \sin^2 x} - \frac{b(2a + 1) \cos x}{2 \sin^2 x} \); 
3) \( U(x) = \frac{b^2 - a(a + 1)}{2 \sin^2 x} + \frac{b(2a + 1) \cos x}{2 \sin^2 x} \); 
4) \( U(x) = \frac{a(a - 1) + b^2}{2 \sin^2 x} - \frac{b(2a - 1) \cos x}{2 \sin^2 x} \).

Their spectra,

1) \( E_n = -(b - a - 2n)^2/2 \); 
2) \( E_n = -(a - n)^2/2 \); 
3) \( E_n = -(a - n)^2/2 \); 
4) \( E_n = (a + n)^2/2 \),

are found by analogy with the procedure used in Sec. 4.

Whether Eq. (7) has solutions other than those in (8) remains an open question. It may be that the "shape invariance" of the potentials, which is expressed by Eqs. (3) and (7), is also a necessary condition for the possibility in principle of finding the exact spectrum.

Finally, we note that the energy of the ground state can be calculated for the potential \( U(x) = (W^2 - W')/2 + C \) with the derivative \( U(x) \) provided that \( \psi = \exp(-\int W(x'Dx') \) is normalizable. For example, for \( U(x) = (a^2x^6 - 3ax^2)/2 \) we find \( E_0 = \sqrt{a} \) (reckoned from the bottom of the potential well).

There is the interesting possibility that this approach might be generalized to multidimensional cases and to field theory.

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