Pis'ma v ZhETF, vol. 87, iss. 10, pp. 669 – 673

Fractional Quantum Hall Effect and vortex lattices

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Submitted 21 April 2008

It is demonstrated that all observed fractions at moderate Landau level fillings for the quantum Hall effect can be obtained without recourse to the phenomenological concept of composite fermions. The possibility to have the special topologically nontrivial many-electron wave functions is considered. Their group classification indicates the special values of of electron density in the ground states separated by a gap from excited states.

PACS: 73.43.–f

The experimental discovery of Integer Quantum Hall Effect (IQHE) by K.v Klitzing (1980) and Fractional Quantum Hall Effect (FQHE) by Tsui, Stormer and Gossard (1982) was one of the most outstanding achievements in condensed matter physics of the last century.

Despite the fact that more than twenty years have elapsed since the experimental discovery of quantum Hall Effect (QHE), the theory of this phenomenon is far from being complete (see reviews [1, 2]). This is primarily true for the Fractional Quantum Hall Effect (FQHE), which necessitates the electron–electron interaction and can by no means be explained by the one-particle theory, in contrast to the IQHE. The most successful variational many-electron wave function for explaining the 1/3 and other odd inverse fillings was constructed by Laughlin [3, 4]. The explanation of other observed fractions was obtained by various phenomenological hierarchial schemes with construction of the "daughter" states from the basic ones (Haldane 1983, Laughlin 1984, B.Halperin 1984).

In those works, the approximation of extremely high magnetic field was used and all states were constructed from the states at the lowest Landau level. However, this does not conform to the experimental situation, where the cyclotron energy is of the order of the mean energy of electron–electron interaction. Moreover, this approach encounters difficulties in generalizing to the other fractions. Computer simulations also give a rather crude approximation for the realistic multiparticle functions, because the number of particles in the corresponding calculations on modern computers does not exceed several tens.

The most successful phenomenological description is given by the Jain's model of "composite" fermions [5, 6], which predicts the majority of observed fractions. According to this model, electrons are dressed by magnetic-flux quanta with magnetic field concentrated in an infinitely narrow region around each electron. It is assumed that even number of flux quanta provides that these particles are fermions. The inclusion of this additional magnetic field in the formalized theory leads to the so-called Chern–Simons Hamiltonian. This approach is described in details in [7].

However, this theory gives an artificial 6-fermionic interaction whereas the actulal calculations use quite crude mean field approximation of the "effective" magnetic field as the sum of the external magnetic field and some additional artificial field that provides the total magnetic flux quanta in accordance with Jain's model of composite fermions.

In the present work I shall show how to remove some restrictions of Jain-Chern-Simons model and obtain a more general and more simple model which does not change the standard Coulomb interaction of electrons. The main concept is associated with the notion of topological classification of quantum states. There is a number of topological textures in condensed matter physics: Vortex lattices in a rotating superfluid, Abrikosov vortices in superconductors, skyrmions in 2d electron systems at integer fillings of Landau levels. It is difficult to give an exact topological classification of the multiparticle wave function for various physical systems. Possibly the most simple and general definition can be done using canonical transformation of the field operators of the second quantization. The canonical transformation of the field operators is one which does not change their commutation relations. I do not consider the statistical transmutations which possibly can not be achieved at low energies considered in condensed matter physics. In general there must be the proper topological classification of the canonical transformations itself.

In this work I consider the simplest case of the fermion canonical transformation not including spin degrees of freedom and assuming the full polarization of 2d electrons

$$\psi(\mathbf{r}) = e^{i\alpha(\mathbf{r})}\chi, \psi^{+}(\mathbf{r}) = \chi^{+}e^{-i\alpha(\mathbf{r})}$$

with $\alpha(\mathbf{r}\ )$ having vortex kind singularities. It is evident that χ and χ^+ satisfy Fermi kind commutation relations if ψ and ψ^+ satisfy them. Inserting these expressions into the standard hamiltonian for the interacting electrons (with omited spin indices)

$$H = \frac{\hbar^2}{2m}\int \psi^+(-i\nabla - \frac{e}{c\hbar}\mathbf{A})^2\psi d^2r + \quad (1)$$

$$+\int \frac{U(\mathbf{r}-\mathbf{r}')}{2}\psi^+(\mathbf{r})\psi^+(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})d^2rd^2r' \quad (2)$$

we get a new Hamiltonian

$$H = \frac{\hbar^2}{2m}\int \chi^+(-i\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A})^2\chi d^2r + \quad (3)$$

$$+\int \frac{U(|\mathbf{r}-\mathbf{r}'|)}{2}\chi^+(\mathbf{r})\chi^+(\mathbf{r}')\chi(\mathbf{r}')\chi(\mathbf{r})d^2rd^2r' \quad (4)$$

where $U(r)$ is Coulomb interaction. I want to consider a set of periodic vortexlike singularities in $\nabla\alpha$. Vector $\nabla\alpha$ can be expressel in terms of Weierstrass zeta function used in the theory of the rotating superfluids [8] given by the converging series

$$\zeta = \frac{1}{z} + \sum_{T_{nn'}\neq 0}(\frac{1}{z-T_{nn'}} + \frac{1}{T_{nn'}} + \frac{z}{T^2_{nn'}}) \quad (5)$$

where $z = x + iy$ is a complex coordinate on 2d plain, $T_{nn'} = n\tau + n'\tau'$ and τ,τ' are the minimal complex periods [9] of the vortex lattice. The phase factor $e^{i\alpha}$ will be simple function on 2d plain if $\nabla\alpha = K(Re\zeta, Im\zeta)$ and

$$\alpha(\mathbf{r}) = K\int_{\mathbf{r_0}}^{\mathbf{r}}(Re\zeta dx + Im\zeta dy) \quad (6)$$

with integer K of any sign. The quantity K and the periods τ,τ' define the topological class of multiparticle wave function. The transformed Hamiltonian (3) can not be restored to the ininitial form (1) by any smooth finite transformation of the function α . That makes it topologically stable. I shall investigate the pecularities of the ground state and excitations for this model at low temperature.

Having in mind large magnetic fields it is interesting to consider the simplified version of the hamiltonian (3) without the interaction term

$$H' = \frac{\hbar^2}{2m}\int \chi^+[-i\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A}(\mathbf{r})]^2\chi d^2r \quad (7)$$

This Hamiltonian has properties very close to the Hamiltonian with a constant magnetic field. Indeed the translation on any period $\boldsymbol{\tau}$ of the vortex lattice gives an additional constant in the brackets

$$\mathbf{r}\to\mathbf{r}+\boldsymbol{\tau}$$

$$[-i\nabla + \nabla\alpha - \frac{e}{c\hbar}A(\mathbf{r})] \to$$

$$[-i\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A}(\mathbf{r}) + \boldsymbol{\delta}(\boldsymbol{\tau}) - \frac{e}{c\hbar}\mathbf{A}(\boldsymbol{\tau})] \quad (8)$$

due to the properties of Weierstrass function $\zeta(z+\tau) = \zeta(z)+\delta(\tau)$ and the linear dependence of the external vector potential $\mathbf{A}(\mathbf{r})$ at constant magnetic field. The additonal constant terms can be removed by the gauge transformation of the field operators χ,χ^+. Thus the proper magnetic translation does not change Hamiltonian (7).

If we introduce the "effective" vector potential $\mathbf{A}_{\text{eff}} = \mathbf{A} - \frac{c\hbar}{e}\nabla\alpha$, the magnetic translation is given by the transformation

$$T_m(\boldsymbol{\tau})\chi = \chi(\mathbf{r}+\boldsymbol{\tau})\exp(\frac{ie}{c\hbar}\mathbf{A}_{\text{eff}}(\boldsymbol{\tau})\mathbf{r}) \quad (9)$$

for any real period of the vortex lattice.

It is easy to connect $\mathbf{A}_{eff}(\boldsymbol{\tau})$ with the "effective" magnetic flux through the unit cell of the vortex lattice given by the contour along it's boundaries

$$\Phi = \oint \mathbf{A}_{\text{eff}}d\mathbf{r} = \mathbf{A}_{\text{eff}}(\boldsymbol{\tau}_1)\boldsymbol{\tau}_2 - \mathbf{A}_{\text{eff}}(\boldsymbol{\tau}_2)\boldsymbol{\tau}_1.$$

On the other hand it can be calculated directly using the definition of $\mathbf{A}_{\text{eff}}$

$$\Phi = \mathbf{B_0}\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2 + K\Phi_0 \quad (10)$$

where $\Phi_0 = 2\pi\frac{e}{c\hbar}$ is the quantum of the flux, B_o is the external magnetic field.

As was shown by E.Brown (1964) [10], J.Zak (1964) [11] (see also [12]) the simple finite representation of the ray group of magnetic translations can be obtained only for rational number of the flux quanta per unit cell

$$\Phi = \frac{l}{N}\Phi_0 = B_0 s + K\Phi_0 \quad (11)$$

where s is the area of the unit cell of the vortex lattice, l and N are integers without common factors.

Thus the situation for the vortex lattices is isomorphous to the case of uniform magnetic field with a rational number of the flux quanta per the unit cell. Therefore it is possible to use all the argumentation following the paper [10] in constructing of the finite representation for the ray group of magnetic translations. In order to construct the finite representation one must impose certain boundary conditions on the solutions of Schroedinger equation with the hamiltonian (7). The simplest is the magnetic periodicity

$$T_m(\mathbf{L})\chi(\mathbf{r}) = \chi(\mathbf{r}) \quad (12)$$

where $\mathbf{L} = \mathbf{L_1}, \mathbf{L_2}$ define the size of the sample, $\mathbf{L_1} = NM_1\boldsymbol{\tau}_1, \mathbf{L_2} = NM_2\boldsymbol{\tau}_2$ with integer M_1, M_2. It easy to show that any magnetically translated function χ according to (9) will also satisfy (12). The simplest realization is the vortex lattice consisting of exactly $N \times N$ unit cells.

This conditions is the analog of Born-von Karman conditions in the absence of magnetic field. Indeed in a large enough system the density of states practically does not depend on the exact form of boundary conditions. But the restriction to the finite representations is important.

The matrices of the representation are

$$D_{jk}(0) = \delta_{jk}$$
$$D_{jk}(\boldsymbol{\tau}_1) = \delta_{jk} \exp i(j-1)\frac{l}{N}$$
$$D_{j,k}(\boldsymbol{\tau}_2) = \delta_{j,k-1}$$
$$(\mathrm{mod}\, N)(j,k = 1,2...N) \quad (13)$$

and the general matrix of the representation

$$D_{jk}(n_1\boldsymbol{\tau}_1 + n_2\boldsymbol{\tau}_2) =$$
$$= \exp i\pi\frac{ln_1}{N}[n_2 + 2(j-1)]\delta_{j,k-n_2}(\mathrm{mod}\, N) \quad (14)$$

The traces of all matrices are zero except identity which has a trace equak to N. The sum of the squares of traces is N^2 . Therefore the representation is irreducible . The square of the dimensionality is also N^2 therefore there can be no other nonequivalent representation. The dimensionality of the representation gives N fold degeneracy of the energy levels for Hamiltonian (7). The number of the equivalent representations in a regular representation is also N. These equivalent representations correspond to the states with different energies for the real "crystal" containing not only vortices but also a periodic potential. But for the simplyfied Hamiltonian (7) all N^2 translations are on equal footing and do not change the energy of the state because they commute with the hamiltonian but do not commute with the each other. Therefore all N^2 elements of the regular representation must have the same energy.

If $|l| \neq 1$ there is a possibility to have l different periodic solutions, corresponding to the different number of the zeros for the wave function inside the magnetic cell having l flux quanta, that is analogous to the unit cell with l places for electrons in an ordinary crystal without magnetic field. That gives the additional energy levels. But the number of the states for the given energy level is one per each unit cell of the vortex lattice. Thus Hamiltonian (7) corresponds to an "empty" lattice with N^2 states with the same energy. The spectrum of this Hamiltonian has no equidistant energy levels that is valid only for the oscillator problem.

The limitation to the single magnetic cell $N\boldsymbol{\tau}_1, N\boldsymbol{\tau}_2$ can be easy removed by the consideration of the vortex lattices with dimensions $N_1\boldsymbol{\tau}_1, N_2\boldsymbol{\tau}_2$ where $N_1 = NM_1$, $N_2 = NM_2$ for integer M_1,M_2. The representations of the larger group of $N_1 \times N_2$ operations can be formed from the already discussed.

For this group there are M_1M_2 representation of dimensionality N. The matrices corresponding to the translation $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ differ from already given only by a phase factor. These representations can be labelled by a vector with reciprocal space components of q_1,q_2

$$D^{\mathbf{q}}(\boldsymbol{\tau}_j) \equiv \exp(-iq_j\tau_j)D(\boldsymbol{\tau}_j); j = 1,2 \quad (15)$$

where possible values of $\mathbf{q}_j$ are given by

$$q_j = \frac{2\pi C_j}{N_j\tau_j}$$
$$j = 1,2$$
$$C_1 = 0,1,...M_1 - 1;\ C_2 = 0,1,...M_2 - 1 \quad (16)$$

Thus each representation corresponding to a given value of $\mathbf{q}$ is N dimensional and one has N equivalent representations in a regular one. The total number of the states is $M_1M_2N^2$ for the regular representation, also corresponds to one state per each unit cell of the vortex lattice. By the construction every magnetic translation does not change the Hamiltonian (7) and therefore all states of the regular representation corresponds to the same energy. Therefore the spectrum is discrete and nonequdistant. If $|l| \neq 1$ where will be $|l|$ additional levels corresponding to $|l|$ zeros of the wave function inside the magnetic unit cell.

At large magnetic fields the Hamiltonian (7) will be dominating in the full Hamiltonian (3) because it linearly depends on magnetic field while the interaction term is proportional to the square root of it. In this case the energy of the ground state including the interaction can be obtained by the perturbation theory

$$E_0 = M_1M_2N^2\epsilon_0 +$$
$$\frac{1}{2}\int U_c(|\mathbf{r}-\mathbf{r}'|) < \chi^+(\mathbf{r})\chi^+(\mathbf{r}')\chi(\mathbf{r}')\chi(\mathbf{r}) > d^2rd^2r' \quad (17)$$

here the angle brackets denote the average over the Slater determinant of the fully filled ground state with the energy ϵ_0 of the Hamiltonian (7). The energy gap dividing the ground state from the next discrete level with the energy ϵ_1 at large magnetic fields must be proportional to the value of the external magnetic field. In the performed experiments [13] the linear dependence of the jump for electron chemical potential in strong magnetic fields was observed for the fractions 1/3 and 2/3. The full expression for the gap must be obtained by the numerical calculation of any Bloch function for the given representation and is dependent on K, N,l and periods τ_i

One can see that in the model of the vortex lattices the gap does not depend exclusively on the interaction term like it was suggested in most of theoretical works based on the degeneracy of the ground Landau level. Opposite , it is almost independent from the interaction in strong magnetic fields. The resolution of this paradox is probably the same as in the rotating superfluid. The origin of the observed vortex lattices in a rotating superfluid is connected with the thermodynamic energy in the rotating frame $E' = E - \Omega \mathbf{M}$,where Ω is the angular velocity and $\mathbf{M}$ is the angle momentum of the superfluid. That requires the superfluid velocity to be equal to the velocity of the solid body rotation and the vortex lattice is a good approximation in a superfluid. Really it is connected with a different dependence of the energy on the size of the system giving the preference to the solid body rotation irrespective to the microscopical internal structure of the superfluid.

The case of magnetic field differs a bit from the case of the rotation for a superfluid. The quantization of the orbital motion gives rise to Landau diamagnetism i.e. to the increase of the system energy due to the appearance of magnetic field. It is possible to reduce this effect by the vortices with the opposite sign of the flux.

The previous group analysis valid for a rational number of the flux quanta show that the energy gaps are opened at the special electron densities corresponding to one electron per each unit cell of the vortex lattice, that gives according to Eq.(11) the electron density

$$n_e = \frac{B}{\Phi_0} \frac{N}{l - NK}. \tag{18}$$

The occurrence of any specific numbers of vortex flux quanta can be dictated by the ground-state energy. The observed fractions in FQHE correspond to the following tables

$$K = -2, \ \ l = 1$$

N	1	2	3	-5	-2	-3	-4	4	∞
ν	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{4}{9}$	$\frac{1}{2}$

That fractions correspond to celebrated Jain's rule [6]. Half filling of the Landau level $n_e = \frac{B}{2\phi_0}$ in the external field corresponds to a vanishingly small effective magnetic field (zero number of flux quanta per elementary cell).

Other observed fractions correspond to

$$K = -1, \ \ l = 1$$

N	-4	4	2
ν	$\frac{4}{3}$	$\frac{4}{5}$	$\frac{2}{3}$

where one has double of the fraction 2/3, and

$$K = -1, \ \ l = 2$$

N	-7	-5	5	2
ν	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{5}{7}$	$\frac{1}{2}$

here one has not observed double of the fraction 1/2 with the gap ($B_{\text{eff}} \neq 0$).

Thus, I have reproduced the key statement of the Jain's theory of composite fermions and obtained the explanation of practically all observed fractions at moderate Landau levels filling in an unified frame without any hierarchial schemes. Of course, these results are quite crude and, in some points hypothetical. The energy gap, the properties of elementary charge and collective excitations, and the conductivity calculations, as well as the analysis of different K and l, N values are still open questions. The approach to these problems needs some extensive numerical calculations. The preliminary results where published in [14]. The degeneracy of the ground state in a periodic magnetic field was established previously for Pauli equation [15].

I am grateful to V.G. Dolgopolov, V.F. Gantmakher, V.B. Timofeev, V.I. Marchenko and M.V. Feigelman for helpful discussions. This work was supported by the Russian Foundation for Basic Research ,the Program of the Presidium RAS, and the Program of OFN RAS. the Program for Supporting Scientific Schools.

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