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On the Formation of Freak Waves on the Surface of Deep Water

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Numerical simulation of the fully nonlinear water equations demonstrates existence of giant breathers on surface of deep water. Numerics show that this breather (or soliton of envelope) does not loose energy. Existence of such breather can explain appearance of freak waves.

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Introduction. Formation of giant freak waves from relatively calm sea is one of the most enigmatic phenomena not only in physical oceanography but in the whole physics of nonlinear waves also. At the moment a vast literature is devoted to explanation to this effect. However practically all papers on this subject have the same weak point. They use for description of wave trains one another form of "envelope equation" based on expansion in powers of a small parameter, nonlinear Shredinger equation for deep water derived in [1]:

$$\epsilon = \lambda/L.$$

Here λ is the wave length, L is a width of the wave pulse. However, in reality $\epsilon \simeq 1$, and the freak wave is a single solitude event. So far offered theories can not explain this effect.

In the present article we formulate the following suggestion. We consider that the freak wave is the "giant breather" Another words, this is a solution of the Euler equation which is periodic in a certain moving reference frame. We support this hypothesis by numerical experiment. We would like to stress that we solve not envelop but exact fully nonlinear Euler equation for potential flow of ideal incompressible fluid with free surface. We study only two-dimensional fluid – one coordinate is horizontal, another is vertical.

Question of existence of breathers in the model a little beyond the "envelope equation" (namely in the Dysthe equation [4, 5])was studied numerically in [2, 3] and [6]. Here we study fully nonlinear regime of potential flow.

1.Basic equations. We assume that the fluid fills the area

$$-\infty < y < \eta(x,t), \qquad -\infty < x < \infty.$$

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The velocity field is potential one

$$V = \nabla\phi, \qquad \nabla\cdot V = 0, \qquad \triangle\phi = 0.$$

Hydrodynamic potential $\phi(x,y,t)$ and shape of the surface $\eta(x,t)$ satisfy the equations

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g\eta = -\frac{P}{\rho}, \quad &&\text{at } y = \eta,\\ &\frac{\partial\eta}{\partial t} + \eta_x\phi_x = \phi_y, &&\text{at } y = \eta. \end{aligned} \tag{1}$$

One can perform the conformal transformation to map the domain that is filled with fluid on Z-plane $(Z = x + iy)$ the lower half-plane

$$-\infty < u < -\infty, \quad -\infty < v < 0, \quad w = u + iv, v < 0$$

in w-plane. This transformation is realized by function

$$Z = Z(W), \qquad Z = x + iy.$$

Potential $\phi(x,y)$ transforms to the complex velocity potential $\Phi(W,t)$. Both of them, Z and Φ are analytic functions in the lower half-plane.

Following [7] one can introduce following variables

$$R = \frac{1}{Z_w}, \qquad \text{and } V = i\Phi_z = i\frac{\Phi_w}{Z_w}. \tag{2}$$

In new variables the Euler equation reads

$$\begin{aligned} R_t &= i(UR_w - RU_w),\\ V_t &= i(UV_w - RB_w) + g(R-1). \end{aligned} \tag{3}$$

Here U and B are

$$\begin{aligned} U &= \hat{P}(V\bar{R} + \bar{V}R)\\ B &= \hat{P}(V\bar{V}). \end{aligned} \tag{4}$$

$\hat{P} = \frac{1}{2}(1 + i\hat{H})$ – projector operator. We must stress that

$$R \to 1, V \to 0, \text{at } v \to -\infty.$$

R and V are periodic function of u (or vanishing at $u \to \pm\infty$).

2. Recall NLSE. For weakly nonlinear flows there is well-known equation describing weakly modulated wavetrain, nonlinear Shredinger equation:

$$i(\frac{\partial A}{\partial t} + C_g A_x) - \frac{\omega_0}{8k_0^2}A_{xx} - \frac{1}{2}\omega_0 k_0^2 |A|^2 A = 0. \quad (5)$$

A is the envelope of the surface elevation $\eta(x,t)$, so that

$$\eta(x,t) = \frac{1}{2}(A(x,t)e^{i(\omega_0 t - k_0 x)} + \text{c.c.}) \quad (6)$$

and ω_0 and k_0 – frequency and wavenumber of the carrier (wavetrain). Well-known solution for $A(x,t)$ is the envelope soliton

$$A(x,t) = e^{-i\Lambda^2 t}\frac{\lambda}{\sqrt{2}k_0^2}\frac{1}{\cosh(\lambda(x - C_g t))}, \quad \Lambda^2 = \omega_0\lambda^2/8k_0^2. \quad (7)$$

For the fully nonlinear equations (3) this $A(x,t)$ corresponds to breather.

In this paper we want to study the following question-*Does the similar solution exist for strong nonlinear flow?* Another words, we have tried to find stationary breather solution with very few carrier wave and highest steepness.

It should be mentioned that "breather" (6) is relevant only for weakly nonlinear flows:

$$\frac{\lambda}{k_0} < 0.07.$$

3. Initial conditions for conformal equations. Initial condition for soliton in physical variables is:

$$\eta(x) = \frac{\lambda}{\sqrt{2}k_0^2}\frac{\cos(k_0 x)}{\cosh(\lambda x)}. \quad (8)$$

Conformal mapping satisfies the following equation:

$$y(u) = \eta(u - \hat{H}y(u)). \quad (9)$$

One can suggest the following iterative procedure to solve the equation (9)

$$y^{n+1}(u) = \eta(u - \hat{H}y^n(u)). \quad (10)$$

4. Numerical simulation of giant breather. For initial condition we have used essentially nonlinear breather with the steepness

$$\lambda/k_0 > 0.4.$$

The idea of the numerical experiment is to study the possibility of existence of breathers. We did it in the following manner:

- **Initial condition.** For the initial conditions for the equations (3) we have used breather that came from NLSE approximation. However the parameters of that breather (8) were chosen far beyond the applicability of the NLSE. Simulation was performed in the periodic domain $L = 2\pi$ with $k_0 = 50$. Value of λ varied from 30 to 50.

- **Damping.** This initial NLSE breather radiates excess of energy (or whatever). In our simulation we have been using damping for that radiation. It was done in the following way. First, simulation was performed in the reference system moving with group velocity. This velocity is calculated during simulation, and is adjusted to keep "breather" in the center of domain. This trick allows us to introduce damping close to edges of the domain to get rid of excessive radiation. After some time, when the radiation vanishes, damping is switched off. So, this trick with dampin looks like

$$R_t \to R_t + \gamma\cos^6\left(\frac{u}{2}\right)R, \quad V_t \to R_t + \gamma\cos^6\left(\frac{u}{2}\right)V, \quad (11)$$

  with $\gamma = 0.05$.

- **Long time evolution.** Since damping is switched off, we observe long time evolution of the remain structure. It should me mentioned here that we want to study the most possible nonlinear regime.

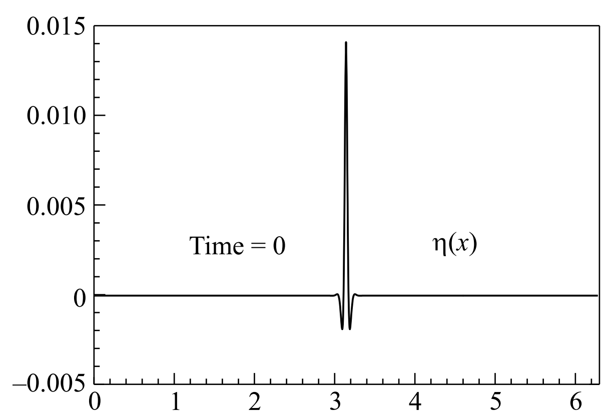

Fig.1. Initial surface as NLSE breather with $k_0 = 50$ and $\lambda = 50$

Below the simulation with $k_0 = 50$ and $\lambda = 50$ is discussed. The initial condition is shown in Fig.1. One can see that under the envelope there are very few wave periods of the carrier.

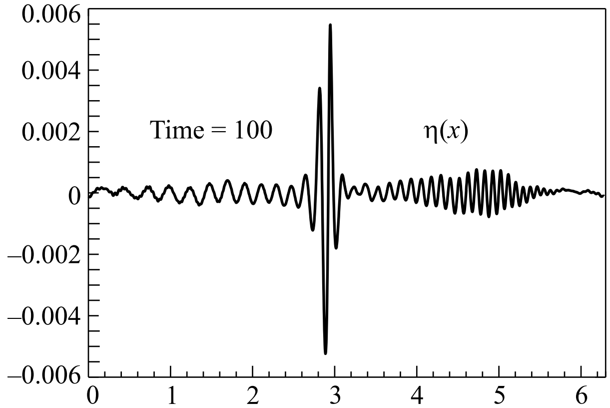

Fig.2. Radiation from the breather

Formally this is exact solution for NLSE. However steepness is so high that NLSE approximation fails. It means that during the evolution some essential changes must happen to this initial condition. After some time initial "hump" radiates during some time that artificial damping acts, see Fig.2. It was switched of at $T = 350$, and surface profile and steepness is shown in Figs.3 and 4. We can think that at this time we start new simulation with the initial conditions as in Fig.3. After more that 2000 wave periods nothing happened, breather propagates with the velocity that approximately 10% large than linear group velocity. Surface profile in shown in Fig.5. Local steepness is shown in Fig.6. It should be pointed that steepness is always less then limiting value $1/\sqrt{3}$, but can be close to this value Looking closely at the steepness as in Fig.7 one can recognize $1/\sqrt{3} \simeq 0.57735$ value for limiting steepness. Those breathers arise from any localized initial conditions, see Fig.8 For example, in the Fig.9 one can see breather with arises from NLSE soliton solution with

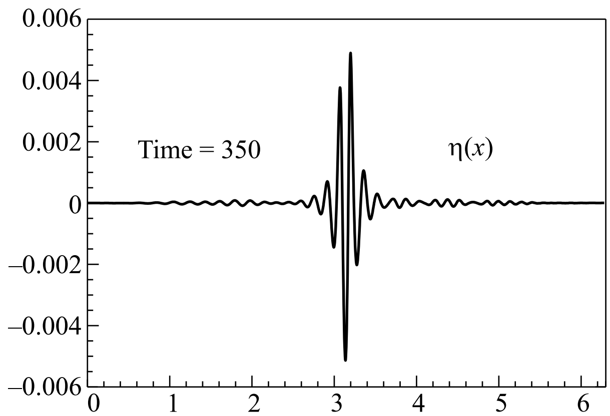

Fig.3. At this moment damping was switched off

$$\lambda/k_0 = 0.6,$$

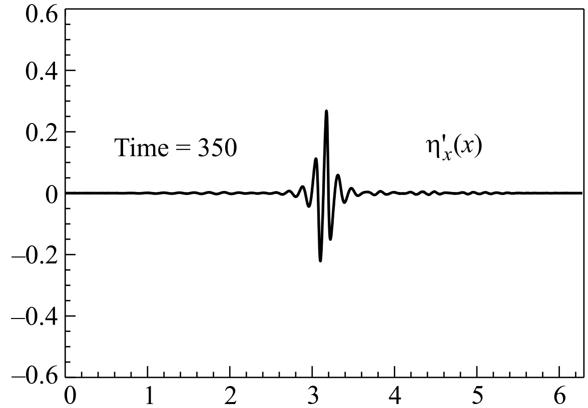

Fig.4. Steepness profile at the moment of switching off damping

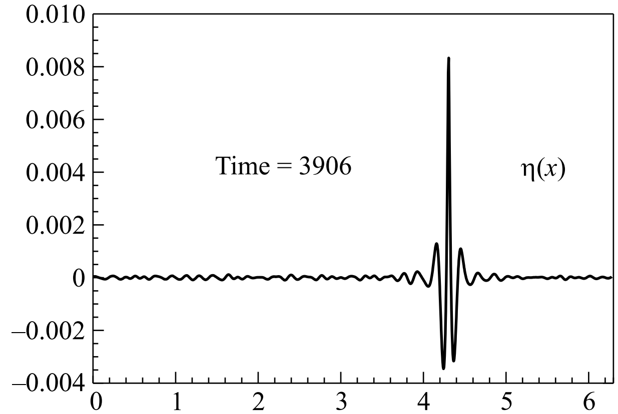

Fig.5. Typical profile of breather

and no damping for radiation. Exess radiation remains in the periodic domain of simulation and does not affect breather. This situation is very similar to the NLS equation which is integrable one.

Hypothesis of integrability is also supported by the picture of the $k-\omega$ spectrum of the breather, see Fig.10.

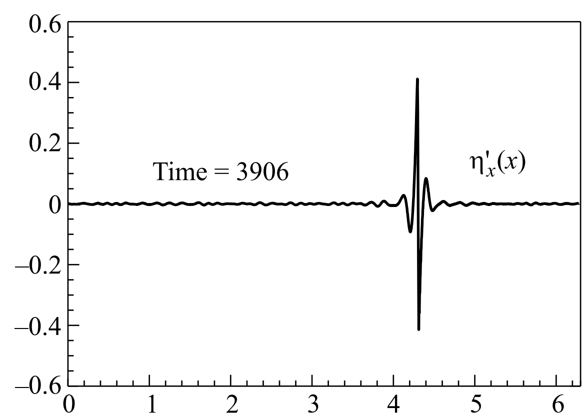

Fig.6. Finall steepness profile

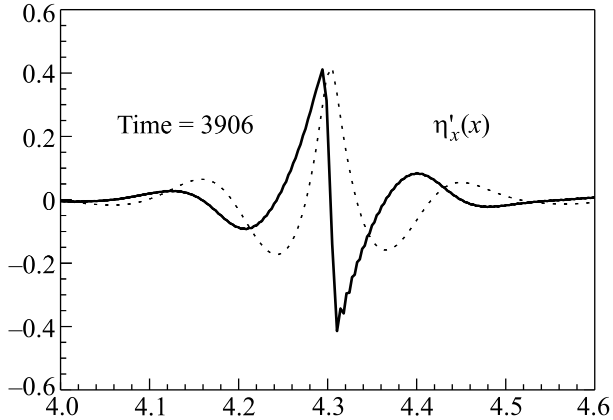

Fig.7. Detailed steepness profile. Dotted line corresponds to the surface profile multiplied by $k_0 = 50$

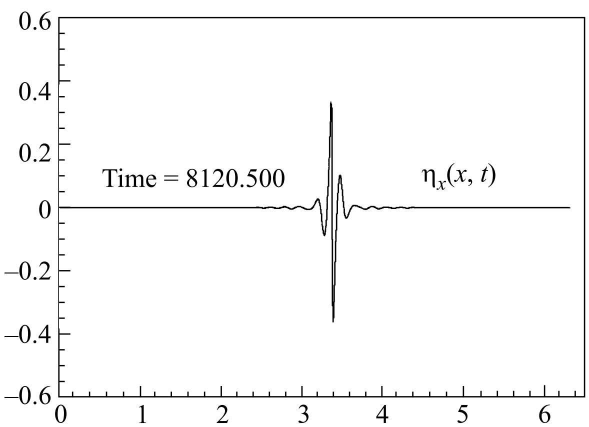

Fig.8. Breather arising from another initial conditions

5. Conclusion. We have shown numerically that strongly nonlinear localized breather can exist on the surface of deep water and propagate for a very long time without lose of energy. We have all reasons to think that existence of such breathers is an explanation of the freak wave phenomenon. We offer to identify freak waves with giant breathers. In this article we do not discuss a problem of how freak waves appear from a relative calm sea. We have to stress that existence of high amplitude breathers is a very exotic phenomenon in the world of nonlinear waves. Such breathers do exist in integrable systems, like nonlinear Shredinger equation. However, in the nonintegrable systems they loose their energy due to radiation [8]. Thus, their existence might be considered as indication of integrability of the Euler equation for potential flow with free surface. But we would like to be more cautions. To make more definite conclusion we have to compare behavior of giant breathers in guaranteed nonintegrable systems, such as MMT-model. These experiments are carried now.

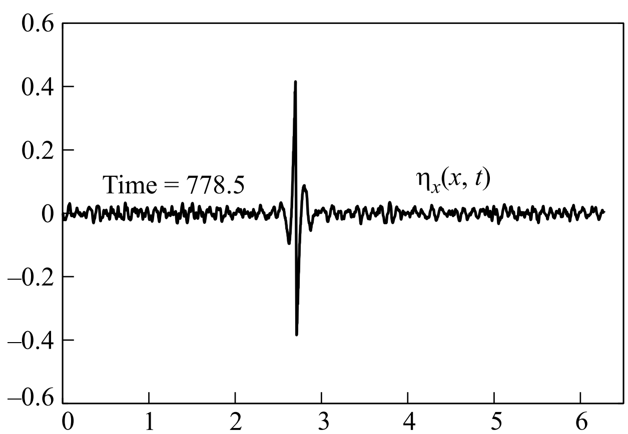

Fig.9. Breather arising from another initial conditions

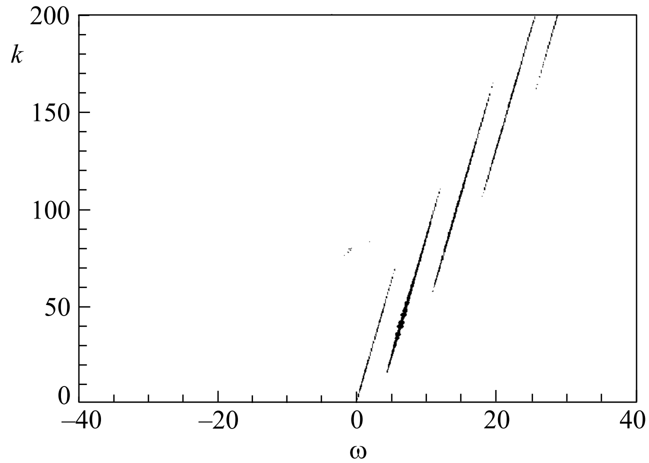

Fig.10. $k - \omega$ spectrum of the breather

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