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The four-loop anomalous dimension of the Konishi operator in $\mathcal{N}=4$ Supersymmetric Yang-Mills theory

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We present the result of a *full direct* component calculation for the planar four-loop anomalous dimension of the Konishi operator in $\mathcal{N}=4$ Supersymmetric Yang-Mills theory. Our result confirms the results obtained from superfield and superstring computations, which take into account finite size corrections to the all-loop asymptotic Bethe ansatz for the integrable models describing the spectrum of the anomalous dimensions of the gauge-invariant operators and the spectrum of the string states in the framework of the gauge/string duality.

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The discovery of the AdS/CFT correspondence [1] with one of its statement about relation between the spectrum of superstrings on $\mathrm{AdS}_5\times\mathrm{S}^5$ and the spectrum of the anomalous dimensions of the gauge-invariant operators in maximally extended $\mathcal{N}=4$ Supersymmetric Yang-Mills (SYM) gauge theory has renewed interest to the perturbative calculations in $\mathcal{N}=4$ SYM model [2]. First such one-loop calculations[2)] were performed for the quasipartonic twist-2 operators with an arbitrary Lorentz spin [4] as a generalization of the famous QCD results [5]. It was found, that eigenvalues of anomalous dimension matrix can be expressed through Ψ-function with shifted argument [4]. This result allowed author to make a suggestion, based on the study of integrability [6] related with Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [7], that the computations of the anomalous dimensions of Wilson operators in $\mathcal{N}=4$ SYM theory should be also related with integrability. The integrability was found by mapping the planar one-loop dilatation operator of $\mathcal{N}=4$ SYM theory into the Hamiltonian of a Heisenberg spin chain [8]. Generalization on the higher-loop orders has allowed to write all-loop asymptotic Bethe ansatz [9], which after adding a dressing factor[3)] [12] stands in the agreement with the Bethe ansatz for the sigma-model from the string side [13].

Some perturbative calculations were performed, which confirmed the results obtained from the integrability. The direct calculation of the two-loop anomalous dimensions matrix for the twist-2 operators with the arbitrary Lorentz spin [14] allowed to confirm the *maximal transcendentality* principle for the matrix eigenvalues [15], which should have the harmonic sums entering into the answer in the given order of perturbative theory. Namely, in the n^{th} order of the perturbative theory the sum of modules of indices in the harmonic sums is equal to $2n-1$. Have this principle in mind one can extract the anomalous dimension for the twist-2 operators with arbitrary Lorentz spin in $\mathcal{N}=4$ SYM theory from the known result in Quantum Chromodynamics (QCD). When the results for the three-loop anomalous dimensions in QCD [16] became available after long time calculations, we easily extracted corresponding result for the three-loop anomalous dimension in $\mathcal{N}=4$ SYM theory [17]. This result was found in a perfect agreement with the results from the asymptotic Bethe ansatz not only for Konishi, but also for twist-2 operators with higher Lorentz spins. The result for the three-loop Konishi was then confirmed by direct calculation [18].

However the asymptotic Bethe ansatz (ABA) can be applied only for the long-range operators but for the short operators, such as Konishi, it will be breakdown by so called "wrapping" effect. Indeed, it was shown [19], that the ABA give at the four loops the result, which is in the contradiction with the predictions coming from the BFKL equation. To take into account "wrapping" effect one should make some additional calculations either in the perturbative theory or in the superstring theory. The

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2) The anomalous dimension of the Konishi multiplet has been computed at one (level g^2) and two (level g^4) loops through operator product expansion analysis of the four-point function of stress-tensor multiplets [3].

3) Necessity of a modification of the asymptotic Bethe ansatz was obtained by direct calculation of the four-loop MHV-amplitude [10], what allowed to find the large spin limit of the four-loop anomalous dimension. The first nontrivial coefficient for the dressing factor at weak coupling was checked by a direct perturbative evaluation [11].

first one were performed in Ref. [20], where the special class of four-loop diagrams were calculated in the superfield formalism. Superstring calculations were made in Ref. [21] by taking into account so called Lüscher-terms [22]. Both results were obtained for Konishi and now are in the agreement after corrections from the perturbative calculations side. Because both calculations have some of suggestions and use the result obtained from the ABA, it will be nice to have as a check the same result from the full direct calculations of the four-loop anomalous dimension of the Konishi operator. In this paper we have presented the result of the *full* calculation for the planar four-loop anomalous dimension of the Konishi operator in $\mathcal{N}=4$ SYM theory directly in the components, using Laporta algorithm [23], which based on the resolution of the integration by part (IBP) identities and was successfully applied for the calculation of the four-loop renormalization constants in QCD [24], with the method from Refs. [25].

The Konishi operator is the most simple operator in the $\mathcal{N}=4$ SYM theory and is nothing then the kinetic term of the chiral matter superfields [26]:

$$\mathcal{O}_{\mathcal{K}} = \text{tr}\; e^{g_{YM}V}\bar{\Phi}_I e^{-g_{YM}V}\Phi^I, \quad g=\sqrt{\lambda}/4\pi, \qquad (1)$$

where $\lambda = g_{YM}^2 N_c$ is the t'Hooft coupling constant. Its low lying state in components is:

$$\mathcal{O}_{\mathcal{K}} = \text{tr}\;\bar{\phi}_i\phi^i = \text{tr}\left[A_iA^i + B_iB^i\right], \qquad i=1,2,3\,, \quad (2)$$

where ϕ^i is the complex adjoint scalar field, while A^i and B^i are the real adjoint scalar and pseudoscalar fields correspondingly.

The calculation of the anomalous dimension of such operator in an usual way [5] will rise the infrared divergencies (see Refs. [27] about the same problem for the similar operator $A^\mu A_\mu$). To avoid its one should allow to flow a momentum through the operator and calculate the vertex type diagrams instead of the initial propagator type diagrams. However we can nullifying one of the momenta flowing through one of the external scalar field without appearance of any IR divergencies. This trick, based on the Infrared Rearrangement [28] or in more general on the R*-operation [29][4)], is used in the calculation of the renormalization constants of the vertices and its application for the calculation of the anomalous dimension of the similar operator $A^\mu A_\mu$ can be found in Refs. [27]. After nullifying one of the external momenta we effectively obtain again the propagator type diagrams, which can be easily evaluated by FORM [30] package MINCER [31] up to three loops with the method from Refs. [32]. However for the four-loop calculations MINCER is still unavailable but we can use the method of Laporta [23]. To use this method for our four-loop calculations we following Refs. [25] expand all propagators on the external momentum to make the divergence of the diagram logarithmic, then nullifying external momentum in propagators (only in denominators of corresponding Feynman integrals, not in numerators) and make all lines massive with the same mass. In this way we obtain massive tadpole diagrams instead of initial massless propagator type diagrams. In the case of our calculations all diagrams have already logarithmic divergence and we do not need expansion at all. For obtained tadpoles we apply the method of Laporta [23], which is based on the resolution of the IBP identities and has some of implementations in the different languages/CAS such as AIR [33], DiaGen/IdSolver [34] and FIRE [35]. We have used our own implementation of the Laporta's algorithm [23] in the form of the MATHEMATICA package BAMBA [36] with the master integrals from Ref. [24]. Our realization is very close to AIR [33] with some improvements, major from which is the usage of the symmetry of the tadpole integrals[5)].

For the dealing with a huge number of diagrams we have used a program DIANA [37], which call QGRAF [38] to generate all diagrams. We have written few routines, which allowed us considerable simplified a work with the topologies.

As important check of our program we reproduced the part of the anomalous dimension of the gluon field [24] coming from the pure Yang-Mills gauge theory because the diagrams, which give contributions to this part, have all possible topologies in the four-loop order for the propagator type diagrams.

The calculations were performed with dimensional reduction (DR) scheme [39] and the Feynman rules from Refs. [2, 40], which were used for the calculation of the β-function in the $\mathcal{N}=4$ SYM theory up to three loops [40]. In principal, DR-scheme should be violated in higher-loop orders [41], but we have found[6)] [44], that at least up to three loops the DR-scheme works correctly for the calculations of the renormalization constants of the triple vertices on the contrary of the result from Refs. [42]. This our result [44] allows to hope, that the DR-scheme should work in the four loops at least for the calculations of the propagator type diagrams.

A total number of four-loop diagrams is 131015. All calculations were performed in the Feynman gauge with

4) See Ref. [24] for details.

5) Results for integrals can be obtained under request.

6) A fact, that the result of Refs. [42] is incorrect was pointed out firstly in Ref. [43].

FORM [30], using FORM package COLOR [45] for evaluation of the color traces. For the renormalization we have used the renormalization constants from Ref. [44]. In addition, we need the counterterms for the gluon and scalar "masses":

$$Z^{3-\mathrm{loop}}_{m_g}=\frac{36}{\epsilon}g^2+\left(-\frac{603}{4\epsilon^2}+\frac{111}{4\epsilon}\right)g^4+\left(\frac{977}{4\epsilon^3}+\frac{27335}{48\epsilon^2}+\right.$$
$$\left.+\frac{1}{\epsilon}\left[\frac{43745}{96}-\frac{1539}{2}S_2-\frac{651}{8}\zeta(2)-\frac{217}{2}\zeta(3)\right]\right)g^6,\quad (3)$$
$$Z^{3-loop}_{m_\phi}=-\frac{36}{\epsilon}g^2+\left(\frac{60}{\epsilon^2}+\frac{387}{2\epsilon}\right)g^4+\left(\frac{5155}{12\epsilon^3}-\frac{4603}{12\epsilon^2}+\right.$$
$$\left.+\frac{1}{\epsilon}\left[\frac{11543}{4}+504S_2-\frac{231}{4}\zeta(2)-235\zeta(3)\right]\right)g^6,\quad (4)$$

with S_2 from Ref. [24]. Origin of these counterterms is rather simple. When one calculate the inverse gluon or scalar propagators with the method, which we have used, the result consists of two terms. The first term is proportional to the square of the external momentum and this is the renormalization constant of the gluon (or scalar) field. The second term is proportional to the mass, which was introduced for the IR-regularization of the scalar integrals. This means, that when we will make the renormalization in the next orders of the perturbative theory we should subtract exactly the same two structures both for the gluon and scalar propagators.

The final result after subtraction of the anomalous dimension for the scalar field is:

$$\gamma^{4-\mathrm{loop}}_{\mathcal{K}}=12\,g^2-48\,g^4+336\,g^6+$$
$$+\Big(-2496+576\,\zeta(3)-1440\,\zeta(5)\Big)\,g^8\qquad (5)$$

in a full agreement with the results of Refs. [20, 21].

An important check of our result (5) is the absence of some special numbers such as $\zeta(2)$, $\zeta(4)$, S_2 and other, which enter in the scalar master integrals from Ref. [24].

As a by product we have obtained the following result for the planar four-loop anomalous dimension of the scalar field in the planar limit:

$$\gamma^{4-\mathrm{loop}}_{\phi}=4g^2-2g^4+\left(\frac{23}{2}-27\,\zeta(3)\right)g^6+$$
$$+\left(\frac{1669}{24}+\frac{423}{4}\zeta(3)+\frac{57}{4}\zeta(4)-290\zeta(5)\right)g^8.\qquad (6)$$

Our result of the *full direct* component calculation for the planar four-loop anomalous dimension of the Konishi operator in $\mathcal{N}=4$ SYM theory (5) confirms that indeed the additional contribution to the anomalous dimension of the operators coming from the "wrapping" effect can be calculated both from the perturbative theory [20] and from the superstring theory [21] by taking into account the corrections to ABA from the additional diagrams or from the Lüscher-terms correspondingly. We want to stress that our calculations don't contain any suggestions at all and can be considered as unique "experimental" check for the computations from Refs. [20, 21] including the correctness of the asymptotic Bethe ansatz for the planar AdS/CFT system.

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