# INSTABILITY AND SELF-FOCUSING OF SOLITONS IN THE BOUNDARY LAYER

*A.N.D'yachenko and E.A.Kuznetsov*

*L.D.Landau Institute for Theoretical Physics, Moscow 117334, Russia*

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**It is demonstrated both analytically and numerically that in the limit of small viscosity the instability of one-dimensional long-wave solitons in the boundary layer results in their self-focusing and collapse. Theoretical predictions are in a qualitative agreement with the experiments [1].**

**1.** In the nonlinear acoustics there exists some analog of the self-focusing of light. This is the collapse of acoustic waves with positive dispersion [2] that can be considered as the nonlinear stage of the Kadomtsev - Petviashvili (KP) instability of one-dimensional solitons [3]. Unlike the light self-focusing the mechanism of the KP instability (and, respectively, of the collapse) is connected with decreasing of a soliton velocity with the increase of the pulse amplitude.

In this letter we show that one-dimensional solitons, propagating in the boundary layer and representing holes in the mean velocity profile, undergo the instability of the winding type, analogous to that for acoustic solitons. This instability is of focusing type, leading at the nonlinear stage to the separation of 1D soliton into individual clusters and to the forthcoming self-focusing of each cluster. We show that both theoretical and numerical results presented in this paper are in a qualitative agreement with the experimental data on the coherent structures in the boundary layer [1].

**2.** We consider a shear flow with a constant velocity $U$ parallel to the plane of a wall and depending on the normal coordinate $z$. We will assume that i) function $U$ has no inflection points, ii) $U$ tends to the constant value $U_0$ while $z \to \infty$ and iii) the first derivative $U'$ at $z=0$ is positive, $U'(0) > 0$. The given velocity profile $U(z)$ is stationary only for an ideal fluid. For any sufficiently small viscosity it is not so: the boundary layer becomes turbulent, but the mean value of the velocity $\overline{U(z)}$ behaves in the same way. In this case, however, the boundary layer thickness $h$ depends on the distance $x$ from the edge of the blowing plate and is given by the estimation (see [4]):

$$h \propto \frac{x}{\log \mathrm{Re}},$$

where Re$= U_0h/\nu$ is the Reynolds number. We will consider the low-frequency oscillations (the wave length $\lambda \gg h$ ) insensitive to the high-frequency turbulent fluctuations and neglect the dependence $h$ on $x$ that is correct for large value of $\log \mathrm{Re} \gg 1$ $(x \gg h)$. In this case it is possible from the Euler equation in a weak nonlinear approximation to derive the equation for the amplitude $A(x,y,t)$:

$$\frac{\partial A}{\partial t} + U'(0)AA_x + U_0h\frac{\partial}{\partial x}\hat{k}A = 0 \qquad (1)$$

where $h = U_0/U'(0)$, the Fourier transform of the integral operator $\hat{k}$ is modulus $|\mathbf{k}| = (k_x^2 + k_y^2)^{1/2}$. The amplitude $A$ in the leading order is connected with the velocity fluctuations along the mean flow (parallel to $x$) by means of $v_x \approx A(x,y,t)U'(z)$. Equation (1) was derived first by V.I.Shrira [5] and represents the two-dimensional generalization of the well-known Benjamin-Ono (BO) equation describing long waves in the stratified liquids. It should be noted that, for this problem, this equation in 1D case was derived first in [6] taking into account a small viscosity . The equation (1) is written with the accuracy of $(kh)^3$. In the next order the imaginary contribution appears to the frequency $\omega = U_0 h k_x |k|$ providing a weak damping.

3. By the usual rescaling equation (1) can be written in the standard form of the BO equation:

$$u_t = \frac{\partial}{\partial x}\hat{k}u - 6uu_x = \frac{\partial}{\partial x}\frac{\delta H}{\delta u} \tag{2}$$

where the Hamiltonian is

$$H = \frac{1}{2}I_1 - I_2 \quad (I_1 = \int u\hat{k}u d\mathbf{r}, \quad I_2 = \int u^3 d\mathbf{r}).$$

Soliton solutions of (2) travelling along $x$, $u = u_s(x - Vt, y)$, represent stationary points of $H$ for fixed $x$−projection of the momentum $P_x = 1/2\int u^2 d\mathbf{r}$,

$$\delta(H - VP_x) = 0 \qquad \text{or} - Vu_s - \hat{k}u_s + 3u_s^2 = 0. \tag{3}$$

For 1D case the solution of Eq.(3) can be found explicitly:

$$u_s = 2V/3(x^2V^2 + 1)^{-1} \quad (V > 0). \tag{4}$$

In 2D case Eq.(3) has a more broad class of solutions. We will be interested only in the ground soliton as a solution of (3) without nodes (it was found in [7] numerically). According to the Lyapunov theorem, solitons will be stable if they realize the minimum of $H$. Under scaling transformations, remaining $P_x$, $u_s(\mathbf{r}) \to (1/a)^{d/2}u_s(\mathbf{r}/a)$, $H$ becomes a function of the scaling parameter $a$:

$$H(a) = \frac{I_{1s}}{2a} - \frac{I_{2s}}{a^{d/2}} \tag{5}$$

(here and below the subscript $s$ denotes values of the integrals on the soliton solution). Hence, one can see that in 1D case $H(a)$ has a minimum corresponding to the soliton. To show that 1D soliton realizes the precise minimum of $H$ against all perturbations, let us use the inequality [8]:

$$\int u^3 d\mathbf{r} \le C\left(\int u\hat{k}u d\mathbf{r}\right)^{d/2}\left(\int u^2 d\mathbf{r}\right)^{\frac{3-d}{2}} \tag{6}$$

where constant $C = I_{2s}I_{2s}^{-d/2}(2P_{xs})^{\frac{d-3}{2}}$ Hence, substituting estimation (6) into $H$ and putting $P_x = P_{xs}$, in 1D case one arrives at the following inequality for $H$,

$$H \le H_s + 1/2(I_1^{1/2} - I_{1s}^{1/2})^2, \tag{7}$$

that becomes precise on 1D soliton.

4 For both (1D and 2D) cases solitons move in the upstream direction opposite where linear waves propagate: $v_{gr}V < 0$. They represent holes in the mean velocity profile and therefore move slowly while increasing their amplitude. Thus, for a soliton weakly modulated in transverse direction, the regions with a small amplitude overtake those with a large amplitude. It obviously leads to an instability of the self-focusing type for the soliton front. This instability is analogous to the Kadomtsev - Petviashvili instability of the acoustic solitons[3]. The growth rate for the instability can be easily determined in a long-wave limit. Omitting all calculations (they are similar to that in [9]) we present the final answer for the growth rate:

$$\gamma^2 = \frac{k_y^2 V^2}{2} > 0, \tag{8}$$

where $k_y$ is the wave number of the perturbations. The development of the instability has to result in a soliton separation into individual clusters with a typical size of the order of that for the soliton. There exist several possibilities of the further evolution of each cluster. First of all, the formation of 2D soliton from such a cluster is impossible, because the Hamiltonian on 2D soliton is identically equal to zero (it follows from $\partial_a H|_{a=1} = 0$). Secondly, by the same reason, clusters can not disappear due to the dispersion - the initial state in the form of 1D soliton (plus small perturbations) has a negative $H$, but the broadening distribution has a positive $H$. At the same time, as follows from (5) in 2D case, for states with negative $H$ the Hamiltonian under scaling transformations becomes unbounded from below as $a \to 0$. Therefore one should expect the collapse, which is familiar to the falling down of some particle in unbounded self-consistent potential. Indeed, there is no any alternative to the collapse in the considered case. It is, in particular, confirmed by the following estimation, correct for arbitrary region $\Omega$ with a negative Hamiltonian $H_\Omega$ (compare with [10] and [11]),

$$\max_{x\in\Omega} u \geq \frac{|H_\Omega|}{2P_{x\Omega}}. \tag{9}$$

Hence one can see that $\max u$ as a function of $t$ is always bounded by the conservative value ($H < 0$!). So, the vanishing or yet some sufficient decreasing of the initial maximum are impossible. Moreover, if initially in some separate region $H_\Omega < 0$ and the radiation happens from this region then, due to the radiative waves carrying out positive portions of both $H$ and $P_x$, the Hamiltonian of this region becomes more and more negative and increases its absolute value, but $P_{x\Omega}$ as a positive value decreases. In accordance with (9), it leads to an increase of $\max u$. Thus, the radiation promotes collapse.

The equation (2) in 2D case and two-dimensional nonlinear Schrodinger equation (NLSE) manifest the same critical behavior under the scaling transformations , i.e., the Hamiltonians for both systems under these transformations behave as homogeneous functions of the scaling parameter $a$. Therefore, our system should be called critical similar to 2D NLSE. One should remind also that for the critical NLSE (see, for instance, [13]) the energy (or the number of particles) engaging into singularity during the collapse is finite and the shape of the collapsing cluster coincides with the shape of 2D soliton. By analogy, one should expect the same critical behavior for the considered system. It is easy to see that equation (2) can be satisfied by the self-similar collapsing solution (more exactly - substitution) of the form $u = (t_0 - t)^{-1/2} f(\vec{r}(t_0 - t)^{-1/2})$. On this solution the energy, coinciding up to a constant multiplier with $P_x$, does not depend on $t$ (if the integral converges).

As for the critical NLSE (see, for instance, [12, 13]) it is possible to introduce the notion of the critical power $P_{x,cr} = P_{xs}$ given by the ground soliton and independent on $V$.

Using by inequality (6) at $d = 2$ one can get the following estimation for $H$:

$$H \geq \frac{I_1}{2}\left(1 - \left(\frac{P_x}{P_{x,cr}}\right)^{1/2}\right) \qquad (10)$$

It means that the Hamiltonian for fixed $P_x < P_{x,cr}$ is bounded from below and reaches its lower boundary on the solutions with mean $\langle k \rangle$ tending to zero. For initial conditions with $P_x \leq P_{x,cr}$ the long-time asymptotic state will manifest the distribution broadening due to the dispersion.

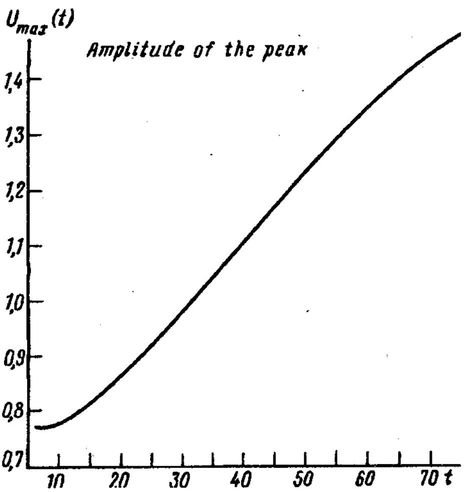


Fig.1

5. To check the proposed theory the straightforward numerical integration of equation (2) was performed. The equation was solved with the help of FFT in the domain $\{-4\pi < x < 4\pi; 0 < y < 4\pi\}$, symmetrical with respect to $y$ $(y \to -y)$, with the boundary conditions:

$$u|_{x=-4\pi} = u|_{y=4\pi} = 0; \qquad \hat{k}u|_{x=4\pi} = 0.$$

Such boundary conditions do not conserve the $x$–momentum as well as the Hamiltonian and permit small amplitude waves to leave the counting region. The initial conditions were chosen in the Lorentzian form with two varying widths $1/V_x$ and $1/V_y$ along $x$ and $y$, respectively,

$$u(\mathrm{r}) = 2/3\frac{|\mathbf{V}|}{(\mathbf{V}\mathrm{r})^2 + 1}.$$

The variation of V allows to change initial values of $H$ from positive to negative. For all initial conditions with negative $H$ we observed the collapse. As one can see from Fig.1 the amplitude of the peak moving with the acceleration increases. The temporal behaviors of the peak velocity and the peak amplitude were very familiar

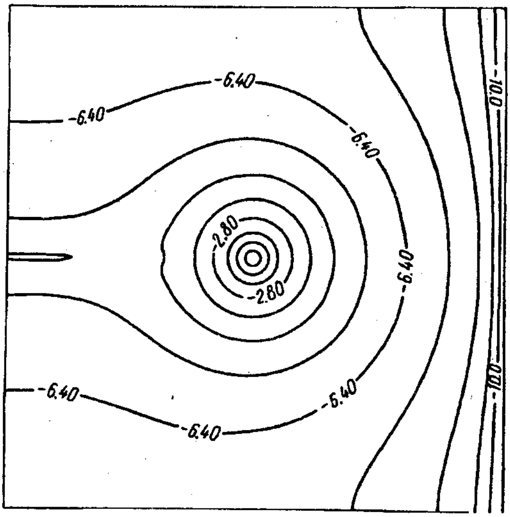


Fig.2

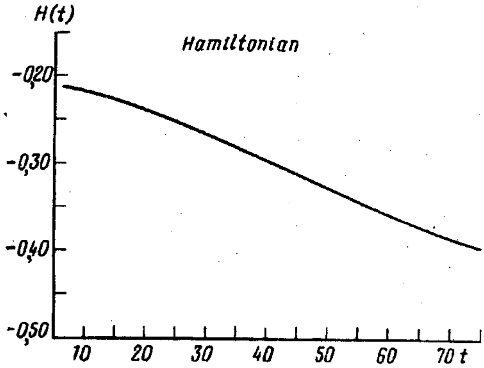


Fig.3

that indicates to the self-similar character of the pulse collapse. While approaching to the singularity, the peak anisotropy vanishes, and the peak distribution becomes nearly symmetric (Fig.2). The $x$–momentum remains approximately constant in each run with a negative $H$. At the same time the Hamiltonian decreases (Fig.3). For the initial conditions with $P_x < P_{x,cr}$ $(H > 0)$ we observe quite slow evolution: the distribution of $u$ at the vicinity of the maximum has the shape familiar to the two-dimensional soliton and slowly decays.

6. In conclusion, we would like to point out some interesting experiments published in [1], summarizing the results of many years of experimental investigations on the onset of the coherent structures in the boundary layer of the blowing plate by the mechanical vibrating system near the edge of the plate (see, also [14]). According to these experimental data at the initial stage the one-dimensional solitons are excited, later (for larger distances from the plate edge) one-dimensional solitons demonstrate the instability resulting, by the authors terminology, in the formation of thorns, i.e., localized three-dimensional coherent structures. For longer distances a self-focusing of the above structures is observed. The further stage of the development of the thorns-solitons lead to the formation of vortices and their forthcoming separation.

The above theory as well as numerical experiments explain all these experimental observations but not the formation of vortices where equation (1) is inapplicable. Up to now it is not clear whether is capable to describe the given experiment quantitatively. Here one should say that the comparison of the theoretical results based on the analysis of the one-dimensional model (1), i.e., in the framework of 1D BO equation [6] showed rather good correspondence with this experiment. So one may hope also to reach a quantitative agreement between the presented 3D theory and the experiment.

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1. Yu.S.Kachanov and O.S.Ryzhov. Sib. Fiz.-Tekh.Journ. N1, 34-52, (1992).
2. E.A.Kuznetsov, S.L.Musher, and A.V.Shafarenko. Pisma v ZhETF **37**, 204, (1983) [JETP Lett. **37**, 241 (1983)].
3. B.B.Kadomtsev and V.I.Petviashvili. Sov.Phys.Dokl. **15** 539 (1970);
   B.B.Kadomtsev. Collective Phenomena in Plasma. Moscow, Nauka, (1976).
4. L.D.Landau and E.M.Lifshits. Fluid Mechanics. Pergamon Press. 1959.
5. V.I.Shrira. Doklady AN SSSR **308**, 732 (1989).
6. V.I.Zhuk and O.S.Ryzhov. Doklady AN SSSR, **263**, 56 (1982); O.S.Ryzhov. Zh.Vych.Mat.i Mat.Fiz. **29**, 1804 (1990).
7. A.A.Abramyan,Ya.A.Stepanyants, and V.I.Shrira. Doklady AN SSSR **327**, 460 (1992).
8. L.Neirenberg. Ann. Sci. Norm. Sup. Pisa. **20**, N 4, 733 (1966).
9. E.A.Kuznetsov and S.K.Turitsyn. ZhETF, **94**, 119 (1988) [Sov.Phys.JETP, **55**, 844 (1982)].
10. V.E.Zakharov. In: Handbook of Plasma Physics, **3**, ed. A.Galeev, R.Sudan (Elsevier, NY, 1984).
11. E.A.Kuznetsov and S.L.Musher. ZhETF, **91**, 1605 (1986) [Sov.Phys.JETP, **64**, 947 (1986)].
12. S.N.Vlasov, V.A.Petrishchev, and V.I.Talanov. Izv.VUZ. Rdiofizika, **12**, 1353 (1970) [Radiophys.Quantum Electron. **14**, 1062 (1974)].
13. S.Dyachenko, A.C.Newell, A.Pushkarev, and V.E.Zakharov. Physica **57D**, 96 (1992).
14. V.N.Borodulin and Yu.S.Kachanov. Izv. Sib.Branch USSR Ac.Sci. N18, 65 (1988).