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THE LEVEL SPACING STATISTICS IN A FINITE 1D DISORDERED SAMPLE

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The distribution function $\mathcal{P}(\Delta)$ of the spacing Δ between nearest enegry levels is calculated for one-dimensional disordered sample with a finite length L. The evaluation proceeds in terms of the Schroedinger equation with a random potential rather than random matrix ensembles. I consider the common case when a particle's wavelength is small comparing with the mean free path. Thus Δ is expressed in terms of a solution of the equation with a given energy and all the moments $< \Delta^m >$ and then $\mathcal{P}(\Delta)$ are calculated with the use of recently developed functional integral method for 1D random potential problem.

Statistical properties of the level spacing Δ in random quantum systems have been the subject of much investigation from the pioneering works [1–2]. They are also the focus of attention in the recent papers [3]. On the other hand, the results of numerical experiments for chaotic quantum systems [4–6] can be interpretated in terms of quasi-one dimensional quantum mechanics with a random Hamiltonian [7]. Quasi-one dimensional behaviour is shown to be equivalent in many cases to the one in strictly 1D random potential problem with some effective parameters [8–11].

The statistics of Δ in an essentially disordered case has been studied analytically in the thermodynamic limit only. The case of a finite sample is, however, of interest for the physics of mesoscopic systems as well as in the studing of quantum dynamics in a finite-dimensional Hilbert space [6, 12]. In addition, the probability to find small Δ is determined completely by finite-size effects (see below).

In the presen Letter I calculate the distribution function $\mathcal{P}(\Delta)$ for a Schroedinger particle placed on the finite 1D interval $(-L, L)$. The potential $U(x)$ in the particle Hamiltonian $\hat{\mathcal{H}} = -d^2/dx^2 + U(x)$ is supposed to be a random function of x obeying the white-noise Gaussian statistics: $< U(x)U(x') >= D\delta(x - x')$. The result will be obtained in the "fast-phase" limit $kL \gg 1$, $kl \gg 1$, where k is the particle's momentum and $l = 4k^2/D$ is the localization length. The relationship between l and L is arbitrary.

I use here essentially the results and notations of the paper [13] where the new functional integral approach to the 1D random potential problem has been developed.

The eigenfunction $\psi(x)$ of $\hat{\mathcal{H}}$ is the solution of the equation $(\hat{\mathcal{H}} - k^2)\psi(x) = 0$ obeying some conditions in the points $x = -L$ and $x = L$, e.g. $\psi(-L) = \psi(L) = 0$. Let us consider the solution $u_k(x)$ of the Caushy problem $(\hat{\mathcal{H}} - k^2)u_k(x) = 0$, $u_k(-L) = 0$, $u'_k(-L) = 1$. If we represent $u_k(x)$ as $a(x)\sin\phi_k(x)$ then in the fast phase limit mentioned above the level spacing is equal to:

$$\Delta = \frac{2\pi k}{|\partial_k \phi_k(L)|}. \tag{1}$$

(There is no summation over k in this formula). Indeed, excluding the free motion we see that the phase $\phi_k(L)$ can be written as $\phi_k(L) = 2kL + \Phi_s(L/l)$, where the contribution $\Phi_s(L/l)$ is due to the potential and depends on the parameters of the problem via the ratio L/l only. This term as well as its derivative with respect to k are not small by themselves. However, the next derivatives of Φ_s with respect to k have the additional factor $1/kl$ comparing with $\partial_k\Phi_s$ and can be neglected. Requiring the variation of the phase between two nearest levels to be equal to 2π we come to the formula (1). With the same precision it leads to the expression of Δ in terms of $u_k(x)$:

$$\Delta = 2\pi \frac{(u_k')^2 + k^2 u_k^2}{u_k \partial_k u_k' - u_k' \partial_k u_k} \Big|_{x=L} = \frac{2\pi k v_1(L) v_2(L)}{\int_{-L}^{L} v_1(y) v_2(y)\, dy} \; . \tag{2}$$

Here $u_k' \equiv \partial_x u_k$ and the "plane wave components" $v_{2,1}(x) = e^{\pm ikx}\,(u_k'(x) \pm iku_k(x))$ are introduced. The formalism developed in the paper [13] allows us to represent the moments $< \Delta^m >, m \geq 1$, as quantum mechanical matrix elements:

$$< \Delta^m > = \left(\frac{\pi k l}{2}\right)^m \frac{1}{\Gamma(m)} \langle e^{\xi/2} | e^{-2L\hat{H}} | e^{-(m+1/2)\xi} \rangle, \tag{3}$$

where ξ is the coordinate of this 1D quantum mechanics and $\hat{H}$ has the form:

$$\hat{H} = -\frac{1}{l}\partial_\xi^2 + \frac{1}{4l}e^{-\xi} + \frac{1}{4l}. \tag{4}$$

The brackets $< \ldots | \ldots | \ldots >$ in the right hand side of (3) and below denote usual scalar product: $< f_1(\xi) | \hat{A} | f_2(\xi) > = \int_{-\infty}^{+\infty} d\xi\, f_1(\xi) \hat{A} f_2(\xi)$ where $f_{1,2}(\xi)$ are some functions and $\hat{A}$ is some operator. From a given set of moments we can restore immediately the Laplace transform $P(s)$ of the distribution function $\mathcal{P}(\Delta)$. Using the integral representation (formula 8.315 in [14]) of $1/\Gamma(m)$ we come to the expression:

$$P(s) = \sum_{m=0}^{\infty} \frac{(-s)^m}{m!} < \Delta^m > = 1 + \langle e^{\xi/2} | e^{-2L\hat{H}} | \Upsilon(\xi, s) \rangle. \tag{5}$$

where

$$\Upsilon(\xi, s) = \frac{e}{2\pi} e^{-\xi/2} \int_{-\infty}^{+\infty} dt\, e^{it} \exp\left(-\frac{\pi k s}{2l(1+it)} e^{-\xi}\right).$$

The matrix element in (5) can be evaluated noting that

$$\frac{1}{l} e^{\xi/2} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu} \cosh \pi\nu \left(K_{2i\nu+1}(le^{-\xi/2}) - K_{2i\nu-1}(le^{-\xi/2}) \right), \tag{6}$$

and

$$\hat{H} K_{2i\nu\pm1}(le^{-\xi/2}) = \frac{1}{l}(\nu^2 \mp i\nu) K_{2i\nu\pm1}(le^{-\xi/2}). \tag{7}$$

The last equality means that (6) represents the function $e^{\xi/2}$ a a linear combination of eigenfunctions of $\hat{H}$. [1)] Substituting (6), (7) into (5) and performing the inverse Laplace transform we obtain after some manipulations:

$$\mathcal{P}(\Delta) = \frac{l}{k\pi^3}\sqrt{\frac{\Delta l}{8k}} \int_{-\infty}^{+\infty} d\tau \cosh\tau \exp\left(-\frac{\Delta l}{2\pi k}\cosh^2\tau\right) \int_{-\infty}^{+\infty} d\nu \frac{\sin(2\nu L/l)}{\nu} \times \tag{8}$$

$$\times \cosh\pi\nu \exp\left(-2\frac{L}{l}\nu^2 + 2i\nu\tau\right).$$

In deriving (8) the integral representation (8.432 in [14]) of the function $K_\mu(z)$ was used. In the limit $L \to \infty$ for a given Δ the expression (8) is reduced to the well known Poisson distribution [15]:

$$\mathcal{P}(\Delta) = \frac{l}{2k\pi}\exp\left(-\frac{\Delta l}{2k\pi}\right). \tag{9}$$

Finite-size corrections to (9) have order of magnitude $\sim \exp(-L/2l)$. When $\Delta \to 0$ and $L \sim l$ the asymptotics of the function (8) has the form:

$$\mathcal{P}(\Delta) \approx \frac{l}{8k}\sqrt{\frac{l}{2\pi L}} \exp\left[-\frac{l}{8L}\left(\ln\frac{8\pi k}{\Delta l} - \frac{2L}{l}\right)^2\right] \mathcal{F}\left(\frac{l}{2L}\ln\frac{8\pi k}{\Delta l}\right), \tag{10}$$

where the function $\mathcal{F}(x)$ is equal to

$$\mathcal{F}(x) = \sqrt{\frac{1}{x}} \exp\left(x(1-\ln x) - \frac{l}{8L}(\ln x - 1)^2\right). \tag{11}$$

Thus, if $\Delta \to 0$ the distribution function $\mathcal{P}(\Delta)$ goes to zero faster than any power of Δ and cannot be described rigorously by Wigner distribution with any set of parameters. This point differs from results of numerical simulations of quantum chaotical systems [6] and it could be a consequence of topology of the boundary conditions. The logarithmically normal distribution (10) does not correspond, however, to any self-averaging quantity. The lage Δ-tail coincides with the function (9).

The representation (1) becomes exact in the small scattering limit. Thus, the final expression (8) must reproduce in the limit $l \to \infty$ equidistant levels structure. Indeed, changing the integration variable $\nu \to \nu l/L$ we reduce $d\nu d\tau$-integration to saddle points ($\tau = i\pi/2, \nu = \pm\pi/4$) contribution. The latter gives $\mathcal{P}(\Delta) = \delta(\Delta - \pi k/L)$.

It is worth noting that the expectation value of the inverse level spacing $< 1/\Delta >$ calculated by means of the distribution (8) is not affected by localization effects:

$$< \Delta^{-1} > = \frac{L}{\pi k} \tag{12}$$

for an arbitrary l/L.

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[1)] The functions presenting in both sides of (6) are not normalizabile and it cannot be considered as an expansion over a basis in the Hilbert space.

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