

Inflaton field potential producing the exactly flat spectrum of adiabatic perturbations

A. A. Starobinsky

Landau Institute for Theoretical Physics RAS, 119334 Moscow, Russia

Submitted 7 July 2005

Presented is the exact solution of the problem of finding a potential of an inflaton scalar field for which adiabatic perturbations generated during a de Sitter (inflationary) stage in the early Universe have the exactly flat (or, the Harrison-Zeldovich) initial spectrum. This solution lies outside the scope of the slow-roll approximation and higher-order corrections to it. The potential found depends on two arbitrary physical constants, one of those determines an amplitude of perturbations. For small (zero) values of the other constant, a long (infinite) inflationary stage with slow rolling of the inflaton field exists.

PACS: 98.80.–k

Great increase of amount and accuracy of cosmological observational data obtained already and expected in near future makes real the ambitious programme of determination of an initial power spectrum $P_0(k)$ of inhomogeneous density perturbations in the Universe directly from data. Then, if working within the scope of the simplest versions of the inflationary scenario of the early Universe (i.e., with one slowly rolling scalar field ϕ – the inflaton), a natural next step is to reconstruct an effective self-interaction inflaton potential $V(\phi)$ leading to the generation of such $P_0(k)$ from inflaton quantum vacuum fluctuations during a de Sitter (inflationary) stage. Note that in inflationary cosmology, an “initial” power spectrum means the spectrum that had been formed by the end of the inflationary stage, i.e. by the beginning of a post-inflationary power-law evolution of the Universe (it was first calculated in [1] for the $R + R^2$ model [2] and in [3–5] for the “new” inflationary model [6, 7]). Here $k = |\mathbf{k}|$, and the spatial dependence $\exp(i\mathbf{k}\mathbf{r})$ of all Fourier modes of adiabatic (scalar) perturbations is assumed.

From the mathematical point of view, since quantum generation of perturbations reduces to a kind of scattering problem, the reconstruction of $V(\phi)$ may be considered as a specific inverse scattering problem. Its solution is known in the slow-roll approximation for the inflaton field, including first [8], second [9] and third [10] order corrections to this basic approximation with respect to small slow-roll parameters, and in the so-called general slow-roll approximation [11, 12]. It is non-unique if additional information about the spectrum of primordial gravitational waves generated during a de Sitter (inflationary) stage [13] is not used. Namely, generally there exists a one-parameter family of $V(\phi)$ producing the same $P_0(k)$ (here and below I don't count

one more trivial free parameter corresponding to invariance with respect to a constant shift of the inflaton field: $\phi \rightarrow \phi + \phi_0$). However, an exact solution of the reconstruction problem is desirable for at least two purposes:

1) to investigate how good is the convergence of perturbation series in powers of slow-roll parameters (if it takes place at all);

2) to determine the exact degree of degeneracy of the problem, i.e. to find the measure of a set of potentials producing the same perturbation spectrum.

In particular, the problem of accuracy of the slow-roll approximation prediction for $P_0(k)$ (including higher order corrections) has been intensively and critically studied recently using different methods: [14], [15] (the uniform approximation), [16] (the improved WKB-approximation) and others.

By an exact solution I mean a solution of the following system of equations for a spatially-flat Friedmann-Robertson-Walker (FRW) background with a scale factor $a(t)$ and scalar (adiabatic) perturbations described by the Mukhanov variable $Q \equiv u/a$:

$$H^2 = \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right), \quad (1)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \quad (2)$$

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{z} \frac{d^2 z}{d\eta^2} \right) u_k = 0, \quad (3)$$

obtained without any approximations. Here

$$H = \frac{\dot{a}}{a}, \quad z = \frac{a\dot{\phi}}{H}, \quad \eta = \int \frac{dt}{a(t)}, \quad (4)$$

dot means the derivative with respect to t , $u_k(\eta) \exp(i\mathbf{k}\mathbf{r})$ is the wave function of a Fourier mode of the quantum field u (the c -number multiplying the Fock annihilation operator \hat{a}_k), and $c = \hbar = 1$ is put throughout the paper. The variable Q [17] is equal to $\delta\phi_L + \frac{\dot{\phi}}{H}\Phi$ in the longitudinal gauge (Φ is the quasi-Newtonian gravitational potential), or to $\delta\phi_S - \frac{\dot{\phi}}{6H}(\mu + \lambda)$ in the synchronous gauge (μ and λ are the Lifshits variables). The normalized initial condition for u_k corresponding to the adiabatic vacuum at $t \rightarrow -\infty$ ($\eta \rightarrow -\infty$) is

$$u_k = e^{-ik\eta}/\sqrt{2k}. \quad (5)$$

At late times during an inflationary stage in the super-horizon regime ($k \ll aH$, $\eta \rightarrow 0$),

$$u_k/z = HQ_k/\dot{\phi} \rightarrow \text{const} = \zeta(k) \quad (6)$$

($\zeta = -h/2$ in the notation of [4]).

Then the initial spectrum of adiabatic perturbations for a post-inflationary cosmology in the super-horizon regime is (assuming the absence of non-diagonal pressure components):

$$\begin{aligned} \langle \Phi^2 \rangle &= \left(1 - \frac{H}{a} \int_0^t a dt\right)^2 \langle \zeta^2 \rangle = \\ &= \left(1 - \frac{H}{a} \int_0^t a dt\right)^2 \int P_0(k) \frac{dk}{k}, \quad (7) \\ P_0(k) &= \frac{k^3 \zeta^2(k)}{2\pi^2}. \end{aligned}$$

Here $t = 0$ corresponds to the end of inflation. For historical reasons, the slope n_S of the spectrum is defined with respect to density perturbations in the non-relativistic dark matter + baryon component at the present time, $(\delta\rho)_k = -k^2 \Phi_k/4\pi G a^2$ before integration over d^3k . So, $n_S = 1 + d \ln P_0(k)/d \ln k$. Finally, using the equation $\dot{H} = -4\pi G \dot{\phi}^2$ that follows from Eqs. (1) and (2), Eq. (1) can be recast in the Hamilton-Jacobi form [18]:

$$H^2(\phi) - \frac{H'^2(\phi)}{12\pi G} = \frac{8\pi G}{3} V(\phi), \quad (8)$$

where the prime denotes the derivative with respect to ϕ .

Exact solutions of the inverse problem of reconstruction of $V(\phi)$ given $P_0(k)$ are known for the following two cases only, if not speaking about solutions describing universes collapsing towards a singularity.

1) A power-law perturbation spectrum with the slope $n_S = \text{const} < 1$ [19]. Then

$$V(\phi) \propto H^2(\phi) \propto \exp\left(\pm \sqrt{\frac{16\pi G}{q}} \phi\right), \quad (9)$$

$$a(t) \propto t^q, \quad q = \frac{3 - n_S}{1 - n_S} > 1.$$

This is just the power-law inflation. Considered as a function of $\phi(t)$, H is related to $V(\phi)$ through Eq. (8). Note, however, that this is not the only potential producing the $n_S = \text{const} < 1$ spectrum.

2) The case when no perturbations are generated at all (no real created quanta of the inflaton field) [20]:

$$H(\phi) = H_1 \exp(2\pi G \phi^2), \quad (10)$$

$$V(\phi) = \frac{3H_1^2}{8\pi G} \left(1 - \frac{4\pi G \phi^2}{3}\right) \exp(4\pi G \phi^2).$$

In literature, this case is sometimes incorrectly referred as the potential generating the $n_S = 3$ perturbation spectrum. However, one should not forget that generated perturbations are quantum (even quantum-gravitational) and require renormalization. After subtraction of the vacuum energy $\omega(t)/2 = k/2a(t)$ of each mode, no created fluctuations remain in this case. Moreover, a number of real inflaton quanta generated in each perturbation mode \mathbf{k} should be large, because in the opposite case they may not be interpreted as classical perturbations after the end of inflation (see [21] for a more detailed discussion of this point).

Strictly speaking, there is no exit from inflation for the potential (9), and the potential (10) does not admit a low curvature regime at all. However, in the former case $V(\phi)$ can be deformed such that it reaches zero at a sufficiently large value of ϕ . This will result in a very small change of the perturbation spectrum at present scales of interest that may be safely neglected. Sometimes, the case of a parabolic potential near its maximum $V(\phi) = V_0 - m^2 \phi^2/2$ is mentioned as an exactly soluble case. However, it is not such the one in our terminology since in this case $H(\phi)$ is approximated by the constant value $H_0 = \sqrt{8\pi G V_0/3}$.

In this paper, a family of exact solutions for the case $n_S = 1$ is constructed. It is just the initial spectrum proposed by Harrison and Zeldovich [22], after all, for beauty reasons. Note that it satisfies the most recent CMB data [23, 24]. Let us first consider what follows for this case from the slow-roll approximation. Then, the leading term in the power spectrum reads

$$k^3 \zeta^2(k) \propto \left(V^3/V'^2\right)_{t=t_k}, \quad (11)$$

where t_k is the moment when $k = aH$. It is clear that, to get $n_S = 1$, $V^{3/2}/V'$ should not depend on ϕ . Therefore, $V(\phi) \propto \phi^{-2}$. Note that this solution of the reconstruction problem is unique for a given amplitude of the flat spectrum. This kind of inflation was dubbed intermediate inflation in [25] (see also [26]). Its scale factor behaviour is $a(t) \propto \exp(\text{const} \cdot t^{2/3})$. Once more, it does not have an exit from inflation, so it should be modified at large ϕ . A next order slow-roll correction to this potential was considered in [27].

To obtain an exact solution for $H(\phi)$ and $V(\phi)$ in the case $n_S = 1$, note first that, for

$$\frac{1}{z} \frac{d^2 z}{d\eta^2} = \frac{2}{\eta^2}, \quad (12)$$

Eq. (3) reduces to the equation for a massless scalar field in the de Sitter background and has the solution

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) \quad (13)$$

satisfying the initial condition (5). Let us write the general solution of Eq. (12) in the form

$$z = \frac{B}{|\eta|} \left(1 + \frac{|\eta|^3}{\eta_0^3} \right), \quad \eta < 0, \quad (14)$$

where A , η_0 are constants. The limiting case $\eta_0 \rightarrow 0$, when the first term in brackets may be neglected, is not interesting because it corresponds to a collapsing universe (however, it is “dual” to the case $\eta_0 \rightarrow \infty$ considered below). The power spectrum of the growing perturbation mode is $P_0(k) = 1/4\pi^2 B^2$ and does not depend on η_0 (η_0 appears in the amplitude of the decaying mode only and makes it non-scale-free). Thus, we have got the exactly flat spectrum. Present observational CMB data [24] fix the quantity B with $\approx 10\%$ accuracy:

$$\frac{1}{2\pi B} = 4.8 \cdot 10^{-5} \left(\frac{A}{0.9} \exp(\tau - 0.17) \right)^{1/2}, \quad (15)$$

where A is the quantity introduced in [24] and τ is the optical length after recombination. In this notation, $A = 0.9$ corresponds to the value $A = 4.3 \cdot 10^{-4}$ of the other quantity A introduced in [28] to characterize an amplitude of initial perturbations (and conjectured to lie in the range $(3 - 10) \cdot 10^{-4}$ in that paper).

Since the aim of this paper is to find some exact solution, I will not investigate if there exist other forms of z leading to the $n_S = 1$ spectrum, too. The absence of other solutions for z would immediately follow from scaling arguments if we assume that $u_k \propto k^{-1/2} f(k\eta)$ for all η . However, the latter assumption might not be

necessary. Moreover, I will consider only one particular case of Eq. (14) corresponding to the limit $\eta_0 \rightarrow \infty$.

So, let $z = -B/\eta$. Let us express all quantities of interest as functions of ϕ :

$$\begin{aligned} t &= -4\pi G \int \frac{d\phi}{H'}, \\ \ln a &= \int H(t) dt = -4\pi G \int \frac{H}{H'} d\phi, \\ \eta &= \int \frac{dt}{a(t)} = -4\pi G \int \frac{d\phi}{H'} \exp\left(4\pi G \int \frac{H}{H'} d\phi\right), \\ z &= \frac{\alpha \dot{\phi}}{H} = -\frac{H'}{4\pi G H} \exp\left(-4\pi G \int \frac{H}{H'} d\phi\right). \end{aligned} \quad (16)$$

Equating the last line in Eq. (16) to $-B/\eta$, we get the following equation:

$$\int P(\phi) d\phi = -BHP, \quad P \equiv \frac{4\pi G}{H'} \exp\left(4\pi G \int \frac{H}{H'} d\phi\right). \quad (17)$$

After differentiation, Eq.(17) reduces to $P = -B(HP' + H'P)$, or

$$\frac{4\pi G H^2}{H'} - \frac{H H''}{H'} + H' + \frac{1}{B} = 0. \quad (18)$$

Let us introduce dimensionless variables

$$x = \sqrt{4\pi G} \phi, \quad y = B\sqrt{4\pi G} H, \quad v(x) = \frac{32\pi^2 G^2 B^2}{3} V(\phi). \quad (19)$$

Then, from (8), $v = y^2 - (1/3)(dy/dx)^2$. For these variables, Eq. (18) reads:

$$y \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} \right)^2 + \frac{dy}{dx} + y^2. \quad (20)$$

After dividing by y^2 , the last equation can be integrated to $dy/dx = xy - 1$ (an integration constant is excluded by shifting x , i.e., ϕ). Therefore,

$$y = e^{x^2/2} \left(\int_x^\infty e^{-\tilde{x}^2/2} d\tilde{x} + C \right), \quad (21)$$

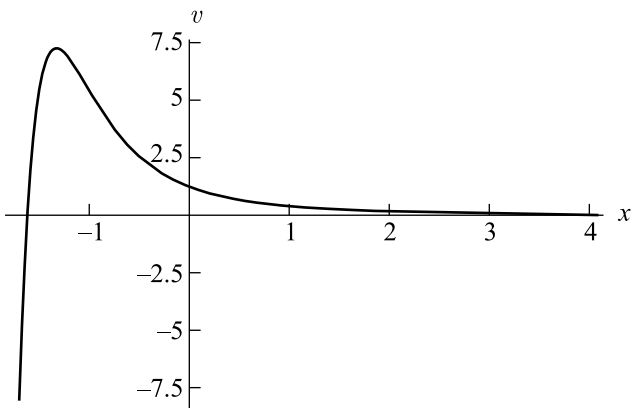
where C is another integration constant. This just yields us a one-parameter family of solutions having $n_S = 1$. The so-called slow-roll parameters for this solution:

$$\begin{aligned} \epsilon(\phi) &\equiv \frac{1}{4\pi G} \frac{H'^2}{H^2} = \left(\frac{1}{y} - x \right)^2, \\ \tilde{\eta}(\phi) &\equiv \frac{1}{4\pi G} \frac{H''}{H} = \frac{1}{y} \frac{d^2 y}{dx^2} = x^2 - \frac{x}{y} + 1. \end{aligned} \quad (22)$$

The partial solution with $C = 0$ has an infinite inflationary stage which is just described by the slow-roll approximation for $x \gg 1$. Its graph is plotted in Figure. Its large- x expansion is

$$y = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \dots, \quad v = \frac{1}{x^2} - \frac{7}{3x^4} + \frac{9}{x^6} - \dots \quad (23)$$

It is straightforward to check that it leads to $n_S = 1$ (as it should be) for the first [8] and second [9] order corrections to the slow-roll approximation. However, these corrections miss the whole 1-parametric family with $C \neq 0$ completely.



The dimensionless potential $v(x)$ for the $C = 0$ case

For $x < 0$, the solution with $C = 0$ has rather peculiar behaviour: the potential $v(x)$ reaches the maximum value $v_{\max} \approx 7.252$ at $x \approx -1.326$, becomes zero at $x \approx -1.618$ and then going to $-\infty$ at $x \rightarrow -\infty$ (however, such effective potentials are considered in string inspired models now). In the latter limit, $y \rightarrow \infty$, so we get an initial curvature singularity at a finite proper time $t_0 < 0$. If $t = 0$ is the moment when $x = 0$ ($v(0) = \pi/2 - 1/3 \approx 1.237$) and the inflationary stage begins, then $|t_0| \sim H^{-1}(0) \sim BG^{1/2}$. The scale factor reaches zero very slowly: $a(t) \propto |\ln(t - t_0)|^{-1/2}$ for $t \rightarrow t_0$. Still the Riemann tensor is not twice integrable for $t \rightarrow t_0$, so this singularity is a strong one. The same refers to all initially expanding ($y > 0$) solutions with $C \neq 0$ and $C > -\sqrt{2\pi}$ – they all begin from such a singularity.

By taking $C < 0$ and very small, it becomes possible to construct a solution with a long but finite inflationary stage. Namely, if $C = -\sqrt{3}x_1^{-2} \exp(-x_1^2/2)$ with $x_1 \gg 1$, then $v(x)$ becomes zero at $x = x_1$ (y still remains $\sim x_1^{-1}$). In this case inflation ends ($\epsilon, |\tilde{\eta}| \sim 1$) at $x = x_1 - \mathcal{O}(x_1^{-1})$. The total number of e-folds is $N_{\text{tot}} = 2\pi G\phi_1^2 = x_1^2/2$. Thus, $C \sim \exp(-N_{\text{tot}})$ that is

in agreement with the general principle that terms not caught by an arbitrary order of a WKB-type expansion are exponentially small. For $x \geq x_1$, one may put $v \equiv 0$. Then the kinetic dominated phase $a(t) \propto t^{1/3}$ follows the inflationary stage. Or, we may assume that v has a local minimum $v = \frac{1}{2}\mu^2(x - x_1)^2$ around this point. It results in oscillations in ϕ and the matter-dominated post-inflationary stage $a(t) \propto t^{2/3}$.

Finally, note that the spectrum of gravitational waves (GW) is not flat for this model: for $1 \ll x \ll x_1$, the tensor-scalar ratio and the slope of the GW initial power spectrum $r = -8n_T = 16/x^2 = 8/N$ where N is the number of e-folds from the *beginning* of inflation. The present upper observational bound $r < 0.36$ [29] requires $N > 22$ for the comoving scale crossing the Hubble radius at present. So, N_{tot} should exceed ~ 70 in this model.

The research was partially supported by the Russian Foundation for Fundamental Research, grant # 05-02-17450, by the Research Programme “Elementary Particles” of the Russian Academy of Sciences and by the scientific school grant # 2338.2003.2 of the Russian Ministry of Education and Science.

1. V. F. Mukhanov and G. V. Chibisov, Pis'ma v ZhETF **33**, 549 (1981) [JETP Lett. **33**, 532 (1981)].
2. A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
3. S. W. Hawking, Phys. Lett. B **115**, 295 (1982).
4. A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982).
5. A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
6. A. D. Linde, Phys. Lett. B **108**, 389 (1982).
7. A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
8. E. D. Stewart and D. H. Lyth, Phys. Lett. B **302**, 171 (1993).
9. E. D. Stewart and J.-O. Gong, Phys. Lett. B **510**, 1 (2001).
10. J. Choe, J.-O. Gong, and E. D. Stewart, JCAP **0407**, 012 (2004).
11. S. Dodelson and E. D. Stewart, Phys. Rev. D **65**, 101301 (2002); E. D. Stewart, Phys. Rev. D **65**, 103508 (2002).
12. M. Joy, E. D. Stewart, J.-O. Gong, and H.-C. Lee, JCAP **0504**, 012 (2005).
13. A. A. Starobinsky, Pis'ma v ZhETF **30**, 719 (1979) [JETP Lett. **30**, 682 (1979)].
14. J. Martin and D. Schwarz, Phys. Rev. D **62**, 103520 (2000); D. J. Schwarz, C. A. Terrero-Escalante, JCAP **0408**, 003 (2004).
15. S. Habib, A. Heinen, K. Heitmann et al., Phys. Rev. D **70**, 083507 (2004); Phys. Rev. D **71**, 043518 (2005).

16. R. Casadio, F. Finelli, M. Luzzi, and G. Venturi, *Phys. Rev. D* **71**, 043517 (2005); preprint gr-qc/0506043 (2005).
17. V. F. Mukhanov, *Pis'ma v ZhETF* **41**, 402 (1985) [*JETP Lett.* **41**, 493 (1985)].
18. A. G. Muslimov, *Class. Quant. Grav.* **7**, 231 (1990); D. S. Salopek and J. R. Bond, *Phys. Rev. D* **42**, 3936 (1990).
19. F. Lucchin and S. Matarrese, *Phys. Rev. D* **32**, 1316 (1985).
20. R. Easther, *Class. Quant. Grav.* **13**, 1775 (1996).
21. D. Polarski and A. A. Starobinsky, *Class. Quant. Grav.* **13**, 377 (1996).
22. E. R. Harrison, *Phys. Rev. D* **1**, 2726 (1970); Ya. B. Zeldovich, *MNRAS* **160**, 1P (1972).
23. A. Benoit, P. Ade, A. Amblard et al., *Astron. Astroph.* **399**, L25 (2003).
24. C. L. Bennett, M. Halpern, G. Hinshaw et al., *Astroph. J. Suppl.* **148**, 1 (2003).
25. J. D. Barrow and A. R. Liddle, *Phys. Rev. D* **47**, 5219 (1993).
26. A. D. Rendall, *Class. Quant. Grav.* **22**, 1655 (2005).
27. A. Vallinotto, E. J. Copeland, E. W. Kolb et al., *Phys. Rev. D* **69**, 103519 (2004).
28. A. A. Starobinsky, *Pisma v Astron. Zh.* **9**, 579 (1983) [*Astron. Lett.* **9**, 302 (1983)].
29. U. Seljak, A. Makarov, P. McDonald et al., *Phys. Rev. D* **71**, 103515 (2005).