

## SINGULARITY OF THE VORTEX DENSITY OF STATES IN *d*-WAVE SUPERCONDUCTORS

*N.B.Kopnin, G.E.Volovik*

*Low Temperature Laboratory, Helsinki University of Technology  
02150 Espoo, Finland*

*L.D. Landau Institute for Theoretical Physics RAS  
117940 Moscow, Russia*

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In *d*-wave superconductors, the electronic density of states (DOS) induced by a vortex exhibits a divergency at low energies:  $N_{\text{vortex}}(E) \sim 1/|E|$ . It is the result of gap nodes in the excitations spectrum outside the vortex core. The heat capacity in two regimes,  $T^2/T_c^2 \gg B/B_{c2}$  and  $T^2/T_c^2 \ll B/B_{c2}$ , is discussed.

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1. Introduction. Two different energy scales, Fermi energy  $E_F$  and the gap amplitude  $\Delta \ll E_F$ , govern the dynamics of fermions in superconductors. This leads to several levels of description of excitations in the inhomogeneous background of the order parameter produced by vortices and textures. (1) A quantum-mechanical approach was used for calculation of the discrete spectrum of bound states in the vortex core [1]. This calculations revealed the existence of the low-energy branch

$$E_0(Q) = -Q\omega_0 \quad (1.1)$$

where the orbital quantum number  $Q$  is half of odd integer for conventional vortices. Here we consider a two-dimensional case, i.e. we neglect the dependence on the momentum  $p_z$  along the vortex axis. The interlevel distance of bound states is small compared to the gap, typically  $\omega_0 \sim \Delta^2/E_F$ .

(2) In the quasiclassical (or Eilenberger) approach, quantum mechanics is applied only for the motion along the trajectory. The energy spectrum is characterized by the continuous impact parameter  $b$  (or the continuous orbital quantum number  $Q = p_F b$ ) and by the trajectory direction (angle  $\alpha$ ) in  $x, y$  plane [2]. The low-energy part of the spectrum corresponds to the chiral branch in Eq.(1.1):

$$E_0(Q, \alpha) = -Q\omega_0(\alpha). \quad (1.2)$$

In a conventional axisymmetric vortex in *s*-wave superconductors,  $\omega_0$  does not depend on  $\alpha$ . In a *d*-wave superconductor, the gap  $\Delta(\alpha)$  is anisotropic, it has the underlying tetragonal symmetry and exhibits four gap nodes. The typical example which displays these properties is  $\Delta(\alpha) \propto \cos(2\alpha)$ , which has four gap nodes at  $\alpha_0 = (2k+1)\pi/4$  with integer  $k$ . In the vicinity of each gap node  $\Delta(\alpha) \approx \Delta'|_{\alpha_0}(\alpha - \alpha_0)$ . As a result the energy spectrum in Eq.(1.2) has a four-fold symmetry and is given by [3]

$$\omega_0(\alpha) \approx (\alpha - \alpha_0)^2 \frac{(\Delta')^2}{E_F} \ln \frac{1}{|\alpha - \alpha_0|} \quad (1.3)$$

in the vicinity of the gap nodes. Properties of the spectrum in the whole range of  $E$  and  $\alpha$  was discussed in [4, 5] in connection to the scanning tunneling microscopy experiment [6].

The quantum limit of Eq.(1.1) is restored by using the Bohr-Sommerfeld quantization rule for the orbital momentum  $Q$  and the canonically conjugated angle  $\alpha$ :

$$\int_0^{2\pi} d\alpha Q(\alpha, E) = 2\pi(n + \gamma), \quad Q(\alpha, E) = -\frac{E}{\omega_0(\alpha)}, \quad (1.4)$$

where  $n$  is an integer and  $\gamma$  is of order unity. This gives the discrete levels

$$E = -(n + \gamma) \left[ \int \frac{d\alpha}{2\pi} \frac{1}{\omega_0(\alpha)} \right]^{-1}. \quad (1.5)$$

For axisymmetric vortices in a  $s$ -wave superconductor, Eq.(1.1) with  $Q = n + 1/2$  is restored if one chooses  $\gamma = 1/2$ . Actually the choice of  $\gamma$  (i.e. either  $\gamma = 1/2$  or  $\gamma = 0$ ) is dictated by the symmetry of the superconducting state in presence of the vortex [7].

For the  $d$ -wave case of Eq.(1.3), the integral in Eq.(1.4) diverges at angles close to the gap nodes. This indicates a singular behavior of the energy spectrum, which originates from  $1/r$  tail of the superfluid velocity  $v_s(r)$  far from the vortex core.

(3) At large distances from the core, a pure classical approach can be used in which the coordinate  $r$  and the momentum  $p$  of the fermionic quasiparticles are considered as commuting variables. The energy spectrum is given by

$$E = \pm \sqrt{\epsilon_k^2 + \Delta^2(\alpha)} + \mathbf{k} \cdot \mathbf{v}_s(\mathbf{r}), \quad (1.6)$$

where  $\epsilon_k = v_F(k - k_F)$ . This approach has been used for calculations of a non-analytical behavior caused by the gap nodes. In  $^3\text{He-A}$ , the order parameter texture induces the effective  $v_s(\mathbf{k}, \mathbf{r})$  and the point nodes lead to a non-analytical density of the normal component at  $T = 0$  [8]. The same effect of gap nodes resulting in a nonzero DOS in presence of the conventional  $v_s$  was discussed in [9–11]. Due to an inhomogeneous vortex-induced velocity  $v_s(\mathbf{r})$  in the mixed state of a  $d$ -wave superconductor, a non-analytical dependence of DOS on magnetic field has been found:  $N(0) \sim N_F(B/B_{c2})^{1/2}$ , where  $N_F$  is DOS in the normal state [3, 12]. The numerical factor in this dependence has been calculated in [13]. Possible experimental realization of such behavior was discussed in [14–16].

Here we consider the energy-dependent DOS,  $N(E)$ , for an isolated vortex and find that  $N(E) \propto 1/|E|$  at  $E \rightarrow 0$  using both the quasiclassical and classical approaches.

2.  $1/E$  divergence of DOS in the classical approximation. DOS in a homogeneous two-dimensional  $d$ -wave superconductor is

$$N(E) = 2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{2} \delta(E \mp \sqrt{\epsilon_k^2 + \Delta^2(\alpha)}). \quad (2.1)$$

Here the factor 2 accounts for the spin degrees of freedom and the factor  $1/2$  takes care of the double counting of particles and holes with different signs of the square root under the  $\delta$ -function.

For small  $E$ , the main contribution comes from the four gap nodes (see also [11]):

$$N(E) = 2 \frac{m}{\pi \Delta'} |E| = 2N_F \frac{|E|}{\Delta'}, \quad (2.2)$$

where  $\Delta' = \partial_\alpha \Delta$  at the gap node and  $N_F = m/\pi$  is the DOS in the normal state. This leads to a  $T^2$ -dependence of the specific heat whose experimental evidences are discussed in [14, 17].

The vortex contribution comes mainly from the superflow around the vortex and from momenta close to the gap nodes  $\mathbf{k}_\alpha$  [3]. The locations of the gap nodes are not important, however we assume the tetragonal symmetry with  $\mathbf{k}_\alpha = \pm k_F \hat{x}, \pm k_F \hat{y}$  for simplicity. All four nodes give equal contributions, thus the extra vortex-induced DOS is

$$N_{d\text{-vortex}}(E) = N_F \frac{4}{\Delta'} \int d^2r (|E - k_F \hat{x} \cdot \mathbf{v}_s(\mathbf{r})| - |E|). \quad (2.3)$$

The velocity far from the vortex is  $\mathbf{v}_s(\mathbf{r}) = \hat{\phi}(\hbar/2m r)$ . Using the variable  $u = r(2mE/\hbar k_F)$  one has

$$N_{d\text{-vortex}}(E) = N_F \frac{v_F^2}{\Delta' |E|} \int_0^{2\pi} d\phi \int_0^1 du (|u - \cos \phi| - u) = \frac{\pi}{2} N_F \frac{v_F^2}{\Delta' |E|}. \quad (2.4)$$

The characteristic dimension of the region near the vortex which contributes to DOS is

$$r(E) = \hbar k_F / 2m |E|. \quad (2.5)$$

When  $|E|$  decreases, the size  $r(E)$  reaches the intervortex distance  $R_B \sim \xi(B_{c2}/B)^{1/2}$  in the vortex lattice. For lower energies,  $|E|$  should be substituted with  $(\hbar k_F / 2m R_B)$ , and the square-root dependence of the DOS in the lattice of the  $d$ -wave vortices [3] is restored. So the Eq.(2.4) holds for the energy  $E$  in the range  $\Delta'(B/B_{c2})^{1/2} \ll |E| \ll \Delta'$ .

**3. Quasiclassical approximation.** In this approach, the radial motion is quantized. We consider only the chiral energy branch. The two remaining variables are: the impact parameter  $b$  (or the angular momentum  $Q = k_F b$ ) and the angle  $\alpha$  of the direction of linear momentum  $\mathbf{k}$  in the  $x, y$  plane. DOS in this approximation is

$$N_{d\text{-vortex}}(E) = \int \frac{d\alpha dQ}{2\pi} \delta(E - E_0(Q, \alpha)), \quad (3.1)$$

where  $E_0(Q, \alpha) = -Q\omega_0(\alpha)$ . Note one sign under the  $\delta$ -function as compared to Eq. (2.1): now there is only one type of excitations on the chiral branch.

For the  $d$ -wave case where  $\omega_0(\alpha)$  is given by Eq.(1.3), the integral in Eq.(3.1) diverges near the nodes: ( $\int d\alpha/\omega_0(\alpha) \sim \int d\alpha/(\alpha - \alpha_0)^2$ , see [3]). This divergence is related to the large extension of the radial wave function  $\Psi(r, \alpha)$  when  $\alpha$  is close to the direction of the gap nodes. To treat this divergence properly, one must include  $\Psi(r, \alpha)$  into the equation for DOS explicitly:

$$N_{d\text{-vortex}}(E) = \int \frac{d\alpha}{2\pi} \int d^2r |\Psi(r, \alpha)|^2 \delta(E - k_F r \sin(\phi - \alpha) \omega_0(\alpha)). \quad (3.2)$$

Here we introduced the impact parameter  $b = r \sin(\phi - \alpha)$ . The radial wave function at large distances  $r \gg \xi$  in a vicinity of the gap node can be obtained from the

Eilenberger equations [18]:

$$|\Psi(\mathbf{r}, \alpha)|^2 = m\Delta' |\alpha - \alpha_0| \exp\left(-2r \frac{\Delta'}{v_F} |\alpha - \alpha_0|\right). \quad (3.3)$$

Eqs.(3.2), (3.3) solve the problem of the divergence, since they have a wider range of applicability than those in Ref.[3], where the condition of small impact parameter compared to the radial extent of the wave function was used. Under this condition, Eqs.(3.2)-(3.3) indeed transform to Eq.(3.1).

We integrate Eq.(3.2) over  $\phi$  introducing variables  $r = \rho v_F / 2\Delta'$ ,  $\alpha - \alpha_0 = \alpha$  and the parameter  $a = L\Delta' / |E|$  (where  $L = -\ln|\alpha|$ ). Summing over 4 gap nodes, one obtains

$$N_{d\text{-vortex}}(E) = \frac{2\pi v_F^2}{\pi \Delta' |E|} \int_0^\infty d\alpha \alpha \int_0^\infty \rho d\rho e^{-\rho\alpha} \frac{1}{\sqrt{(a\rho\alpha^2)^2 - 1}} \Theta(a\rho\alpha^2 - 1). \quad (3.4)$$

This integral is independent of  $a$  and is equal to  $\pi/2$ , thus

$$N_{d\text{-vortex}}(E) = N_F \pi \frac{v_F^2}{\Delta' |E|}. \quad (3.5)$$

This is two times larger than Eq.(2.4) obtained using classical considerations. We can thus conclude that the classical approach is not able to give the correct numerical factor though it feels the relevant physics and provides a correct order-of-magnitude estimate. The same is known for DOS in  $^3\text{He-A}$  textures: the exact DOS obtained quantum-mechanically in [19] is two times the classical result of [8].

4. Conclusion. The singular behavior of DOS can be seen in the temperature dependence of the specific heat in presence of a vortex lattice. Being averaged over vortices, DOS (per one  $\text{CuO}_2$  superconducting layer) contains the bulk term Eq.(2.2) and the vortex term  $n_L N_{d\text{-vortex}}(E)$ , where  $n_L = B/\Phi_0$  is the flux-line density, and  $\Phi_0$  is the magnetic flux quantum:

$$N(E) = \frac{2p_F}{\pi v_F \Delta'} |E| + \frac{B}{\Phi_0} \frac{v_F p_F}{|E| \Delta'}. \quad (4.1)$$

This results in the specific heat:

$$\begin{aligned} C(T) &= \int_{-\infty}^{\infty} N(E) \cosh^{-2}\left(\frac{E}{2T}\right) \frac{E^2 dE}{4T^2} = \\ &= 18\zeta(3) \frac{p_F}{\pi v_F \Delta'} T^2 + 2 \ln 2 \frac{B}{\Phi_0} \frac{v_F p_F}{\Delta'}, \quad \sqrt{\frac{B}{B_{c2}}} \ll \frac{T}{T_c} \ll 1. \end{aligned} \quad (4.2)$$

The second term in Eq.(4.2) is the temperature-independent linear-in-field correction of order  $N_F T_c B / B_{c2}$  to the dominating field-independent bulk term  $\sim N_F T^2 / T_c$ . Two terms become comparable at the temperature  $T/T_c \sim (B/B_{c2})^{1/2}$ , where the crossover occurs to the square-root behavior  $C(T) \sim T N_F \sqrt{B/B_{c2}}$  of Ref.[3] at lower temperatures,  $T/T_c \ll (B/B_{c2})^{1/2} \ll 1$ .

Both high-temperature and low-temperature asymptotic dependences of the specific heat on the magnetic field,  $N_F T_c B / B_{c2}$  and  $N_F T \sqrt{B/B_{c2}}$ , come due to the velocity field far from the vortex and do not depend on the details of the vortex core structure discussed in [20, 21].

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